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Moon's Problem
(A three-body problem)
An investigation into "Intractability and Uncomputability" in Theoretical Physics
A research project

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Some 3000 years ago the Babylonians used to keep accurate records of Moon's periodicities. They had no image of Moon's motion through the sky, but discovered simple arythmetic relationships just by perusing the long series of numbers on their clay tablets. These were the first historical measurements in Natural Sciences. Some 2000 years ago the Greeks introduced images and stories related to the motion of Sun, Moon and Planets in the sky, thus giving birth to science. Reportedly, they knew four basic periodicities of Moon's motion with five decimals, which means an accuracy of one second of time. Beside the Sun's year of ~ 365days , there is a syderal Moon of 27.32days with respect to fixed stars, a synodal Moon of 29.53days as due to the Earth's transport (Moon's phases), a nodal Moon of 27.21days related to its oscillations with respect to the ecliptic plane (and related to the rotation of its axis with respect to the normal to ecliptic, with a period of 18years), and, finally, there is an anomalous Moon of 27.55days, associated to irregularities in its motion, like the drift of the perigee. The Greeks knew to measure the Earth's radius (Eratosthenes), and distance to Moon and Sun (by parallax and triangulations), by using geometric theories. They have accounted for many periodicities and irregularities in Moon's motion by complicate geometric epicycles superimposed over regular, cyclic trajectories (Ptolemy). Moon's and Sun's eclipses were no mistery for them. Essentially the same methodology was followed by Brahe and Kepler in the 16th and 17th century to pin down their enormously multifarious sky observations. The Greeks discovered also the inclination of Earth's axis against the ecliptic (23.5°) and the precession of equinoxes (26 000years). They have also introduced the three basic reference frames, the local one (topocentric), equatorial and ecliptic. In all these endeavours the science of geometry was instrumental, in explaining such complicate movements as those of the heaven bodies, and in gaining new knowledge of things that are not accessible otherwise.

By the same token, looking for explanations and images, Newton introduced the Law of Gravitation at the end of the 17th century. In the next century it allowed the measurement of Earth's mass (Cavendish), and the inference of the mass of other celestial bodies, through their trajectory, a knowledge that it is hard to see how it could have been gained otherwise.

Newton's theory explains perfectly the two-body motion with the Law of Gravitation, the so-called Kepler's problem, where the orbit is an ellipse with the center of motion in one focus, the radius sweeps equal areas in equal times, and the lengths to the 3rd power are proportional to times to the second power. However, the Planetary System consists of many bodies, and even the three-body problem, like Sun-Earth-Moon, raises difficulties. Newton himself devoted much care to account for Moon's motion, with doubtful success and extremely valuable insights. Perhaps the most valuable one is recognizing the perturbation character of Sun attraction to the Moon in its orbit around the Earth. The issue was pursued further by ad-hoc perturbation computations by Euler, Clairaut, d'Alembert and Laplace. It followed Gauss, Lagrange, Hamilton and Jacobi, on which occasion the Analytical Mechanics was built up. Delauney has published around the middle of the 19th century about 2000 printed pages with Moon's calculations, and Hill introduced in 1877 the rotating frame of the Earth in order to simplify the picture. Almost the entire "mathematical physics" was constructed on this occasion, during these two centuries. Gradually, however, the idea occurred that the 3- or n-body gravitational problem is" intractable", "undecidable", and the Planetary System may be unstable. On the occasion of an international contest of mathematics held in honour of Sweden's King in 1890, Weierstrass formulated the problem of representing the planets motion as a perturbation series. Poincare won the contest, without solving the problem. He showed that the perturbation series of his precursors is plagued with divergencies, though looking like an asymptotic series, vanishing denominators and resonances. He showed that there are instabilities in the motion of three bodies interacting trough gravitational forces, and a sensitive, arbitrary dependence of the motion on initial conditions, all related to non-linearities. In general, such a motion is "chaotic", and, following Poincare, "chaos" was introduced in science in the 20th century, together with "intractability", "uncomputability", "computational irreducibility", "unprovability" and "universal computability". Nevertheless, in 1913 Sundman gave a series for the 3-body problem, though infinitely slowly convergent, and therefore practically useless.

Meanwhile, with the advent of electronic computers, the trend changed, and science was left aside. Modern computers are able to calculate Moon's trajectory with accuracy 10^8, which means a few meters for 384 000km Moon-Earth distance, and 10^{-3}s (Moon's velocity is cca 1km/s, and its position can be detemined with cca 1" accuracy, even by naked eye; compare with 150×10^6km the Earth-Sun distance, and cca 30km/s Earth's velocity).[1] Such computations were used in Apollo program of landing people on the Moon (1960-1980). Nevertheless, we have not yet an explanation of the intricate irregularities of Moon's motion through the sky, and what happened with the perturbation series of its motion.

Basically, the Moon's problem was approached as if it would be integrable, though it is not, as there are no other constants of motion except for energy and the total angular momentum (Bruns-Poincare theorem). Apart from perturbing prescribed orbits (the so-called periodic orbits), and using restrictions in the perturbation series, the issue was treated as if there would be basic frequencies in Fourier series. Such frequencies, however, are to be renormalized at every step in the perturbation theory (a procedure known as Poincare-Linstedt procedure), and, in general, the trajectory must be renormalized every perturbation step. Basically, the issue is a consistent treatment of the perturbation series, relaxing all its parameters. This program was recently initiated[2] by setting up a perturbation series in eccentricities, inclination against the ecliptic and Sun's perturbation. To the lowest orders of the perturbation theory there has been accounted for the four Moon's periodicities.

First, Kepler's problem is treated by a series expansion in eccentricities, the radial motion being described as an infinite series of anharmonic oscillators. The method is generalized to any central-field potential, leading to approximate conditions for closed orbits. Though only Newton's potential and the harmonic-oscillator potential close the orbits rigorously (an observation known as Bertrand's theorem), it is shown that many other potentials may close approximately the orbits, to the extent to which an irrational number is approximated by a rational one. This may be

[1] Earth's mass is $\sim 6\times10^{24}$kg , Sun's mass is $\sim 2\times10^{30}$kg, Moon's mass is $\sim 7\times10^{22}$kg; Earth's orbit eccentricity is ~ 0.017, Moon's orbit eccentricity is ~ 0.055; inclination of Moon's orbit against the ecliptic is $\sim 5^\circ$; the constant of universal gravitation is $G = 6.7 \times 10^{-11} m^3/kg \cdot s^2$.

[2] M. Apostol, J. Theor. Phys. **114-118** (2005).

viewed as the first sign of "chaos" in a gravitational problem, or of "undecidability", "unprovability", and "computational irreducibility" (any computer number is rational). As a by-product, the method was applied to Einstein's concept of energy quantization, such that the quantal spectrum of energy was obtained for the Hydrogen Atom as well as for any other central-field potential.

Then, the Sun-Earth-Moon problem is shown to amount to two Kepler's problems inter-connected by a perturbing interaction (which in its lowest order reads as a quadrupolar potential). The theory of perturbation is then set up, according to standard procedure, allowing for the cancellation of the secular terms by renormalizing the frequencies. The unperturbed (zeroth order) solution is written as a series expansion both in eccentricities and the inclination of Moon's orbit against the ecliptic. The computations are carried out in the ecliptic frame, the transformation to the equatorial frame or the local frame being prescribed. In particular, the transform to the rotating frame of the Earth preserves the relative coordinate $\mathbf{r}$ of Moon-Earth motion, but contributes, as it is well known, the rotating velocity $\boldsymbol{\Omega} \times \mathbf{r}$ to the local velocity, where Ω is Earth's angular velocity. The main result is the renormalization of the bare frequency ω of Moon's rotation by one quarter of the squared ratio of the two basic frequencies (Earth's and Moon's) $\omega \rightarrow \omega' = \omega(1 - \Omega^2/4\omega^2)$, which is associated to the syderal Moon ($\Omega/\omega \simeq 1/13$). The synodal Moon follows simply by taking into account the Earth's rotation ($\omega' - \Omega$). Oscillations of Moon's orbiting plane are derived from the effective frequency given by $\omega^2 + \Omega^2$ in the rotating frame. It leads to the correction $3\Omega^2/4\omega^2$ (known to Newton), and gives the nodal Moon and the precession of the Moon's orbit ($4\omega/3\Omega$ to one year). Finally, the deviation from an ellipse leads to correction $3\Omega^2/2\omega^2$ for the drift of the perigee.

This program is pursued further, first by computing higher-order contributions (in order to ensure five decimals to relevant numbers one has to go up the third order of perturbaton theory in perturbation, and corresponding contributions in eccentricities and orbit inclination).

Then, several interesting problems are to be addressed by this procedure. For instance, the couple Jupiter-Saturn exhibits a 2 : 5 resonance in their periods (Jupiter 12years, Saturn 30years), which can be addressed by the present perturbation method, the perturbation being their own gravitational interaction. Then, the Earth loses cca 6minutes per century in its rotation, so friction would be interesting to tackle in the above procedure. The rest of the Planets may affect Earth motion, possibly by a time-space stochastic perturbation. Moon ever displays the same side to the Earth, which is a pinning down of its rotation (explained by Lagrange by Moon's distortion). This blocking of the phase was pointed out by Gauss, and it is related to the pinning of the charge- and spin-density waves in modern quasi-one-dimensional materials, commensurate and incommensurate phases, order parameter, symmetry breaking and Goldstone modes. Moreover, higher-order multipolar perturbation can also be considered, as well as relativistic corrections, or, finally, the Zeeman effect on the Hydrogen Atom.

Along such outlines an investigation may be configured into the "intractability" and "uncomputability" in Theoretical Physics, particularly into a three-body problem.