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On the "bosonization" in two and three dimensions

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Abstract

It is shown that the "bosonization" of the fermions in two and three dimensions proposed by Castro Neto and Fradkin (see, for instance, A. H. Castro Neto and Eduardo Fradkin, Phys. Rev. Lett. **72** 1393 (1994), Phys. Rev. **49** 10 877 (1994)) is not a bosonization, but rather a "fermionization". It is also shown that the bosonic "coherent" state introduced by these authors is not a coherent state, and the corresponding classical action is chosen arbitrarily, in order to mimick some of the properties of the Fermi liquids; in particular, the connection to the Fermi liquid theory is either lacking, or incorrect.

Few years ago, Castro Neto and Fradkin[1],[2] noticed that the Fourier transforms

$$\rho_{\mathbf{q}}(\mathbf{k}) = c^{+}_{\mathbf{k}-\mathbf{q}/2}c_{\mathbf{k}+\mathbf{q}/2} \tag{1}$$

of the fermion-density operator satisfy boson-like commutation relations

$$\langle FS \mid [\rho_{\mathbf{q}}(\mathbf{k}), \rho_{-\mathbf{q}'}(\mathbf{k}')] \mid FS\rangle = \mathbf{q}\mathbf{v}_{\mathbf{k}}\delta(\mu - \varepsilon_{\mathbf{k}})\delta_{\mathbf{k}\mathbf{k}'\delta\mathbf{q}\mathbf{q}'} \tag{2}$$

in the limit $\mathbf{q} \to 0$, when averaged over the Fermi sea $\mid FS\rangle$; here $\varepsilon_{\mathbf{k}}$ is the one-fermion energy, $\mathbf{v}_{\mathbf{k}} = \partial\varepsilon_{\mathbf{k}}/\partial\mathbf{k}$ is the corresponding velocity (Planck's constant is set equal to one), μ denotes the chemical potential, and the spin labels are omitted for simplicity. This is an old observation, and it was systematically exploited, probably for the first time, by Sawada[3] in 1957. Based on this observation the above authors[1],[2] set about to "bosonize" the fermions in two and three dimensions. To this end, normal-ordered operators

$$a_{\mathbf{q}}(\mathbf{k}) = \rho_{\mathbf{q}}(\mathbf{k})\theta(\mathbf{q}\mathbf{v}_{\mathbf{k}}) + \rho_{-\mathbf{q}}(\mathbf{k})\theta(-\mathbf{q}\mathbf{v}_{\mathbf{k}}) \tag{3}$$

are introduced, where θ is the step function, such as $a_{\mathbf{q}}(\mathbf{k}) \mid FS\rangle = 0$, and boson-like commutation relations

$$\left[a_{\mathbf{q}}(\mathbf{k}), a^{+}_{\mathbf{q}'}(\mathbf{k}')\right] = \left|\mathbf{q}\mathbf{v}_{\mathbf{k}}\right| \delta(\mu - \varepsilon_{\mathbf{k}})\delta_{\mathbf{k}\mathbf{k}'}(\delta_{\mathbf{q}\mathbf{q}'} + \delta_{\mathbf{q},-\mathbf{q}'}) \tag{4}$$

are adopted for these operators, as suggested by (2). However, the boson-like commutation relations (4) are not consistent with the definition given in (1) and (3) since

$$a^{+2}_{\mathbf{q}}(\mathbf{k}) = 0 \tag{5}$$

for $\mathbf{q} \neq 0$. Indeed, one can check easily that

$$\rho^{2}_{\mathbf{q}}(\mathbf{k}) = c^{+}_{\mathbf{k}-\mathbf{q}/2}c_{\mathbf{k}+\mathbf{q}/2}c^{+}_{\mathbf{k}-\mathbf{q}/2}c_{\mathbf{k}+\mathbf{q}/2} = n(\mathbf{k})\delta_{\mathbf{q},0} \quad , \tag{6}$$

where $n(\mathbf{k}) = c^+_{\mathbf{k}} c_{\mathbf{k}}$ is the fermion occupation number. Therefore, the "bosonic" operators $a_{\mathbf{q}}(\mathbf{k}), a^+_{\mathbf{q}}(\mathbf{k})$ are in fact fermionic operators, operating only on two states, $\mid FS\rangle$ and $\mid \mathbf{qk}\rangle = a^+_{\mathbf{q}}(\mathbf{k}) \mid FS\rangle$, and the "bosonization" is actually a "fermionization". Since $a_{\mathbf{q}}(\mathbf{k}) = a_{-\mathbf{q}}(\mathbf{k})$, one may restrict oneself to $\mathbf{qv_k} > 0$, and the boson-like commutation relations (4) may be used (approximately) for $\mathbf{k} \neq \mathbf{k}'$ or $\mathbf{q} \neq \mathbf{q}'$, but not for $\mathbf{k} = \mathbf{k}', \mathbf{q} = \mathbf{q}'$ where the a-operators satisfy fermion-like commutation relations $\left\{a_{\mathbf{q}}(\mathbf{k}), a^+_{\mathbf{q}}(\mathbf{k})\right\} = 1$ (and $a^{+2}_{\mathbf{q}}(\mathbf{k}) = 0$). This would suffice to say that the attempt made in Refs.1 and 2 at "bosonizing" the fermions in two and three dimensions is not a bosonization, but rather a "fermionization".

The situation is different in one dimension, where the bosonization is a genuine one. Indeed, the boson-like operators are defined there by

$$\rho_{1,2\mathbf{q}} = \sum_{\mathbf{k}\sim\pm\mathbf{k}_F} c^+_{\mathbf{k}-\mathbf{q}/2} c_{\mathbf{k}+\mathbf{q}/2} \quad , \tag{7}$$

where $\mathbf{k}_F$ is the Fermi momentum, and it is easy to see that, due to the summation over $\mathbf{k}$,

$$(b^+_{1,2\mathbf{q}})^n \mid FS\rangle \neq 0 \tag{8}$$

for any n, where

$$b_{1\mathbf{q}} = \rho_{1,2\mathbf{q}}(\mathbf{k})\theta(\pm\mathbf{qk}_F) + \rho_{1,2-\mathbf{q}}(\mathbf{k})\theta(\mp\mathbf{qk}_F) \ . \tag{9}$$

One may use boson-like commutation relations for these operators, similar with those given by (2), by averaging their commutators over the Fermi sea, and (8) gives then a boson state for any integer n. One may conclude that this is indeed a genuine bosonization. Moreover, the commutators of the kinetic energy with the boson operators $b^+_{1,2\mathbf{q}}$ does not depend on $\mathbf{k}$ in one dimension, in contrast with the two- and three-dimensional case, so that the low-energy states of the one-dimensional fermions can be described entirely in terms of the boson operators (the interaction energy can always be expressed in terms of b, b^+-type operators).[4] As often emphasized, this is in fact the origin of the non-Fermi liquid behaviour of the fermions in one dimension.

Overlooking the contradiction implied by (4) and (5) Castro Neto and Fradkin[1],[2] proceed to constructing a "coherent" state defined by

$$\mid u_{\mathbf{q}}(\mathbf{k})\rangle = \exp\left[\frac{v_{\mathbf{k}}}{2\left|\mathbf{qv_k}\right|} u_{\mathbf{q}}(\mathbf{k}) a^+_{\mathbf{q}}(\mathbf{k})\right] \mid FS\rangle \ ; \tag{10}$$

in view of (5) this can also be written in various other forms, like, for instance,

$$\mid u_{\mathbf{q}}(\mathbf{k})\rangle = \left[1 + \frac{v_{\mathbf{k}}}{2\left|\mathbf{qv_k}\right|} u_{\mathbf{q}}(\mathbf{k}) a^+_{\mathbf{q}}(\mathbf{k})\right] \mid FS\rangle \ ; \tag{11}$$

obviously, the scalar products are different for these states. Restricting oneself to the low-energy states defined by the a, a^+-operators, one can establish the (over)completeness of both (10) and (11) with a gaussian measure; however, the path-integral lagrangean is different, and in fact it is not unique. Indeed, from (10) one obtains the lagrangean density

$$\begin{aligned} \mathcal{L} \ &= \ \sum_{\mathbf{kq}} \frac{v^2_{\mathbf{k}}\delta(\mu - \varepsilon_{\mathbf{k}})}{2\left|\mathbf{qv_k}\right|} \cdot i u^*_{\mathbf{q}}(\mathbf{k}) \frac{\partial}{\partial t} u_{\mathbf{q}}(\mathbf{k}) - \\ &\quad - \langle\left\{u_{\mathbf{q}}(\mathbf{k})\right\}\left|H\right|\left\{u_{\mathbf{q}}(\mathbf{k})\right\}\rangle / \langle\left\{u_{\mathbf{q}}(\mathbf{k})\right\}\left|\right|\left\{u_{\mathbf{q}}(\mathbf{k})\right\}\rangle \quad , \end{aligned} \tag{12}$$

where H denotes the hamiltonian of the system, while, using (11), general factors of the form

$$\left[1+\frac{v_{\mathbf{k}}^2}{|\mathbf{qv_k}|}\delta(\mu-\varepsilon_{\mathbf{k}})\left|u_{\mathbf{q}}^+(\mathbf{k})\right|^2\right]^{-1} \tag{13}$$

would affect the terms in the lagrangean density (12); they arise from the fact that commutators like $[e^{ua}, a^+] = uCe^{ua}$, where $[a, a^+] = C$, differ from commutators like $[1 + ua, a^+] = uC$. Of course, these inconsistencies, like all the others, originate in the fact that bosonic coherent states of the form given by (11) can not be constructed with fermionic operators. One may conclude, therefore, that (11) is not a coherent state, and, in this respect, the choice (12) made by Castro Neto and Fradkin[1],[2] for the classical lagrangean is arbitrary.

Further on, the lagrangean (12) is used by these authors for a hamiltonian H which is quadratic in the a-operators,

$$H = \frac{1}{2V}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} G_{\mathbf{k},\mathbf{k}'}^{\mathbf{q}} a_{\mathbf{q}}^+(\mathbf{k})a_{\mathbf{q}}(\mathbf{k}') \quad , \tag{14}$$

where V is the volume of the system and

$$G_{\mathbf{k},\mathbf{k}'}^{\mathbf{q}} = \delta_{\mathbf{k},\mathbf{k}'}/N(0) + f_{\mathbf{k}-\mathbf{q},\mathbf{k}'+\mathbf{q}} \ ; \tag{15}$$

in (15) $N(0)$ denotes the density of states at the Fermi surface, and $f_{\mathbf{k}-\mathbf{q},\mathbf{k}'+\mathbf{q}}$ stands for a sort of extension of the quasi-particle scattering amplitude of the Fermi liquid theory (actually, the boson-like commutation relations for the a-operators were derived in the limit $\mathbf{q} \to 0$, so that keeping $\mathbf{q} \neq 0$ in the f-function would be incorrect); the summation in (14) is restricted to $\mathbf{qv_k} > 0$. For this hamiltonian the lagrangean density (12) becomes

$$\begin{aligned}\mathcal{L} \;=\; &\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \frac{v_{\mathbf{k}}^2\delta(\mu-\varepsilon_{\mathbf{k}})}{2\mathbf{qv_k}}\cdot\{u_{\mathbf{q}}^*(\mathbf{k})\cdot[i\delta_{\mathbf{kk}'}\frac{\partial}{\partial t} - \\ &-\frac{\mathbf{qv_k}}{VN(0)}\delta_{\mathbf{kk}'}\delta(\mu-\varepsilon_{\mathbf{k}'}) - \frac{\mathbf{qv_k}v_{\mathbf{k}'}}{Vv_{\mathbf{k}}}f_{\mathbf{k}-\mathbf{q},\mathbf{k}'+\mathbf{q}}\delta(\mu-\varepsilon_{\mathbf{k}'})]u_{\mathbf{q}}(\mathbf{k}')\} \ .\end{aligned} \tag{16}$$

First, we note that the hamiltonian (14) can be diagonalized immediately, and, with the choice (15), one obtains both the particle-hole excitations and the collective modes of a Fermi liquid; secondly, one can see that the lagrangean (16) has precisely been chosen in such a way as to reproduce these excitations in terms of the classical coordinates $u_{\mathbf{q}}(\mathbf{k})$; indeed, the corresponding equations of motion for $u_{\mathbf{q}}(\mathbf{k})$ are identical with the well-known equations of motion for the variations of the fermion occupancy $\delta n_{\mathbf{q}}(\mathbf{k})$ in the Fermi liquid theory; hence, it is hard to see the necessity of introducing the lagrangean (16), and to transcribe the equations of motion in terms of $u_{\mathbf{q}}(\mathbf{k})$'s.[5] Restricting themselves to the particle-hole excitations in (16), *i.e.* without including the f-term, Castro Neto and Fradkin[2] reproduce correctly the specific heat of the Fermi liquid by using the imaginary-time formalism of the path integrals; obviously, this is not surprising, as long as these excitations are correctly described by (16), and as long as the Bose statistics is (correctly) assumed for them, when quantized. Actually, the temperature dependence of the specific heat of the particle-hole excitations does not depend on their statistics, as a consequence of the fact that these excitations are constrained to "live" on the Fermi surface; consequently, one can reproduce the correct answer either way, with a right choice of the density of states, which is precisely what Castro Neto and Fradkin[2] do for Bose statistics. However, even so, the thermodynamic computations, as presented by Castro Neto and Fradkin[2], are at least incomplete, and this can be seen from the corrections brought by the f-term in (16) to the specific heat. Indeed, first we note that these corrections, as computed by these authors (and few others, as well; see, for instance, parts of Ref.7), are incorrect, because they amount to higher-order contributions in powers of $\mathbf{q}$ to the spectrum of the particle–hole excitations, which, on one hand is not correct (as known from the Fermi liquid theory), and, on the other hand, the commutation relations for the a-operators were derived only in the limit of the leading $\mathbf{q}$-contribution. Leaving this aside, we still note, however, that the f-term gives the collective modes, which are sound-like oscillations, and, as such, they would bring, for instance, a T^3-correction to the specific heat (which, actually, is negligible) in three dimensions, where T is the temperature; but it is precisely this correction which is not obtained by these authors, in spite of the fact that they assembled all this formalism in order to describe the collective excitations.

Finally, even if the physics of the elementary excitations of a Fermi liquid would be correctly and completely mimicked within such a formalism, there may still remain to justify the origin of the "bosonized" hamiltonian given by (14) and (15). Castro Neto and Fradkin[2] claim to derive this hamiltonian from a fermionic hamiltonian

$$H = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} c^+_{\mathbf{k}} c_{\mathbf{k}} + \frac{1}{2V} \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} f_{\mathbf{k}-\mathbf{q},\mathbf{k}'+\mathbf{q}} c^+_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2} c^+_{\mathbf{k}'-\mathbf{q}/2} c_{\mathbf{k}'+\mathbf{q}/2} \quad , \tag{17}$$

by the "bosonization" procedure. Leaving aside that this "bosonization" procedure is incorrect, it is hard to say what physical system is described by the hamiltonian (17);[6] it seems to correspond to some dressed, and still interacting, fermions, according to the presence of the effective mass m^* in the velocity $\mathbf{v}_{\mathbf{k}} = \partial\varepsilon_{\mathbf{k}}/\partial\mathbf{k}$, and to the presence of the quasi-particle scattering amplitude f in the "interacting" term. One could rather say that this hamiltonian is arbitrarily chosen again, such as to give the correct frequencies of the particle-hole excitations and the collective modes, in terms of the "bosonized" $\rho_{\mathbf{q}}(\mathbf{k})$-operators. The inconsistency of (17), as describing fermionic quasi-particles, and of its counter-part expressed in the lagrangean (16), is immediately seen from the attempt made by Castro Neto and Fradkin[2] to derive the effective mass. Indeed, the current

$$\mathbf{J}_{\mathbf{k}} = \mathbf{v}_{\mathbf{k}} + \frac{1}{V} \sum_{\mathbf{k}'} f_{\mathbf{k}\mathbf{k}'} \delta(\mu - \varepsilon_{\mathbf{k}'}) \mathbf{v}_{\mathbf{k}'} \tag{18}$$

computed by these authors[2] from the variation of the lagrangean (16) is equal to $\mathbf{k}/m^*$, and not $\mathbf{k}/m$ as these authors, and few others,(see, for instance, some of the papers in Ref.7) incorrectly assume; this is obvious from (17), where the velocity $\mathbf{v}_{\mathbf{k}} = \partial\varepsilon_{\mathbf{k}}/\partial\mathbf{k}$ contains the effective mass m^*. Consequently, this would lead to the trivial equalities $m^* = m^*$, and $f = 0$, as expected, since the effect of the f-"interaction" is already included in the effective mass; if, on the other hand, the connection with the original fermions is pursued, in which case the equality $\mathbf{J}_{\mathbf{k}} = \mathbf{k}/m$ would be justified, then, according to the Fermi liquid theory, the sign of the f-term in (17) should be reversed, which would lead to an incorrect equation for the effective mass, as derived within the present formalism. As a matter of fact, it is by far obvious that one can not derive the effective mass equation by working only with quasi-particless, and it is precisely this connection, between the fermionic quasi-particles and the original, interacting fermions, *i.e.* the Fermi liquid theory, which is lacking in the "bosonized coherent state" formalism of Castro Neto and Fradkin,[1],[2] in spite of their claim that "the Landau theory of the Fermi liquids can be obtained from the formalism";[1] this would be necessary, indeed, in order to have a complete, "bosonic" (or "classicized") theory of fermions in two and three dimensions.

Some of the questions raised here can be found, at least partially, in few other recent works on the subject;[7] where it applies, the criticism made in the present paper is aimed at these works, too.

References

[1] A. H. Castro Neto and Eduardo Fradkin, Phys. Rev. Lett. **72** 1393 (1994).

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[3] K. Sawada, Phys. Rev. **106** 372 (1957); K. Sawada, K. A. Brueckner, N. Fukuda and R. Brout, Phys. Rev. **108** 507 (1957).

[4] These remarks are well emphasized in the classical works on the bosonization in one dimension, as, for instance, S. Tomonaga, Progr. Theor. Phys. **5** 544 (1950), D. C. Mattis and E. H. Lieb, J. Math. Phys. **6** 304 (1965), D. C. Mattis, J. Math. Phys. **15** 609 (1974), A. Luther and I. Peschel, Phys. Rev. **B9** 2911 (1974).

[5] The quantization of $\delta n_{\mathbf{q}}(\mathbf{k})$, and implicitly of $u_{\mathbf{q}}(\mathbf{k})$ in this context, in terms of boson excitations describing particle-hole pairs and collective modes, would be a bosonization, of course, though not one performed with the $\rho_{\mathbf{q}}(\mathbf{k})$-operators; and, obviously, this is not what Castro Neto and Fradkin[1],[2] pursue, or do. Incidentally, we note that these authors[1],[2] try to relate the coordinates $u_{\mathbf{q}}(\mathbf{k})$ to the variations $\delta n(\mathbf{k})$ of the fermion occupancy, and implicitly, to the changes in the shape of the Fermi surface, by writing $\delta n(\mathbf{k}) = \delta(\mu - \varepsilon_{\mathbf{k}}) v_{\mathbf{k}} u(\mathbf{k})$, where $u(\mathbf{k})$ is probably understood as $u_{\mathbf{q}}(\mathbf{k})$, and $\delta n(\mathbf{k})$ as $\delta n_{\mathbf{q}}(\mathbf{k})$, *i.e.* the Fourier transforms of the corresponding position-dependent quantities; however, if so, $u_{\mathbf{q}}(\mathbf{k}) = u^*_{-\mathbf{q}}(\mathbf{k})$, and since the definition of the a-operators requires $u_{\mathbf{q}}(\mathbf{k}) = u_{-\mathbf{q}}(\mathbf{k})$, it follows that $u_{\mathbf{q}}(\mathbf{k})$ are real quantities, and not complex ones, as these authors assume.

[6] *Inter alia*, the hamiltonian (17) is not even hermitian, at least, with the chosen $\mathbf{q}$-dependence of $f_{\mathbf{k}-\mathbf{q},\mathbf{k}'+\mathbf{q}}$.

[7] See, for instance, A. Houghton and B. Marston, Phys. Rev. **B48** 7790 (1993), R. Shankar, Revs. Mod. Phys. **66** 129 (1994), F. D. M. Haldane, Helv. Phys. Acta **65** 152 (1992), G. Baym and C. Pethick, *Landau Fermi-Liquid Theory* (Wiley, New York, 1991), A. Luther, Phys. Rev. **B19** 320 (1979); D. Coffey and K. S. Bedell, Phys. Rev. Lett. **71** 1043 (1993), C. Pethick and G. Carneiro, Phys. Rev. **A7** 304 (1973); etc.