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**A note on the wave equation for a semi-infinite line**

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We have often good reasons to admit that a homogeneous scalar wave equation

$$\omega^2 u + \frac{d^2 u}{dx^2} = 0 \tag{1}$$

written here in a simplified form in one dimension with usual notations may hold for the description of certain wave phenomena. Obviously, the quantities in equation (1) must have a definite meaning, which means that function $u$ must be continuous and differentiable on an open domain. We may be interested for instance in solving equation (1) for $x > 0$. Then, on obvious physical grounds, we must admit that $u = 0$ for $x < 0$, and the natural question arises: what happens for $x = 0$? Of course, we have the lateral limit $u \to u_0$ for $x \to 0$ with $x > 0$ and must notice that equation (1) gives

$$\frac{du}{dx}\mid_{x=0} = u_0^{'} = 0 \tag{2}$$

by integration about $x = 0$, where $u_0^{'}$ is the lateral derivative. Equation (2) provides the so-called "free-end" boundary condition. Obviously, $du/dx$ may be viewed as a generalized "force", for instance an elastic force (per unit length), and the boundary condition (2) tells that the "surface" force at $x = 0$ is vanishing.

We might be interested to add a force to equation (1), in particular a force localized at $x = d$, so we may write

$$\omega^2 u + \frac{d^2 u}{dx^2} = f\delta(x - d) \tag{3}$$

and wish to consider the limit $d \to 0$ as describing a force localized on the surface. Now, in dealing with $\delta(x)$ we must understand that this is a highly-peaked function on $x = 0$, such that $\int dx\delta(x) = 1$, but otherwise continuous and differentiable. Similarly, when using the step function $\theta(x)$, $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$, for which $\delta(x) = \theta'(x)$, we must consider it as an abrupt function, but otherwise continuous and differentiable. It follows that we have to admit that solution $u$ in equation (3) must have a little extension to the *lhs* of $x = 0$, and certainly be defined for $x = 0$, of course as its lateral limit $u_0$; and, moreover, be differentiable for $x = 0$ with derivative $u_0^{'}$. Now, it is easy to see that $u$ should be of the form $u\theta(x)$ which makes equation (3) become

$$\left(\omega^2 u + \frac{d^2 u}{dx^2}\right)\theta(x) + u_0^{'}\delta(x) + u_0\delta'(x) = f\delta(x - d) \ . \tag{4}$$

It folows that equation (3) remains unchanged, while, by integrating around $x = 0$, we get the boundary conditions

$$u_0 = 0 \tag{5}$$

or

$$u_0^{'} = 0 \tag{6}$$

for $d \neq 0$ and

$$u_0^{'} = f \tag{7}$$

for $d = 0$. We can see that we get an additional boundary condition, that of a fixed end, given by equation (5). We can use in general the linear combination $\alpha u_0 + \beta u_0^{'}$, where $\alpha$, $\beta$ are constants, for accounting for both boundary conditions (free end and fixed end). In addition, we may see that the surface force $u_0^{'}$ must compensate the force $f$ localized on the surface.

Let us solve equation (3). By Fourier transform it is easy to get the particular solution

$$u_p = \frac{f}{2\omega} \sin \omega (x - d) sgn(x - d) \ . \tag{8}$$

Of course, this particular solution does not satisfy any boundary conditions as given above. In order to do this we must add the solution of the homogeneous equation to this particular solution. We may call it the free solution, as being given by "free waves", in contrast with the particular solution, which corresponds to "forced waves". The free solution reads

$$u_f = A \cos \omega x + B \sin \omega x \ \ , \tag{9}$$

where $A$, $B$ are constants. Therefore, the complete, or general, solution is given by

$$u = A \cos \omega x + B \sin \omega x + \frac{f}{2\omega} \sin \omega (x - d) sgn(x - d) \ . \tag{10}$$

We may impose now the boundary condition, for instance the free-end condition. We get

$$u = A \cos \omega x + \frac{f}{2\omega} \cos \omega d \sin \omega x + \frac{f}{2\omega} \sin \omega (x - d) sgn(x - d) \ . \tag{11}$$

It is interesting to note that the forced waves generates free waves, such as the surface force

$$u_{p0}^{'} = -\frac{f}{2\omega} \cos \omega d \tag{12}$$

generated by the forced waves be compensated by the surface force

$$u_{f0}^{'} = B\omega \tag{13}$$

generated by the free waves, in order to satisfy the free-end boundary conditions.

As a matter of fact, we can get the general solution (11) by taking two free solutions, one for $x < d$ another for $x > d$, and imposing upon them the continuity and the "boundary condition" arising from equaling the jump in their derivative at $x = d$ with the localized force $f$, according to the equation. This can be seen easily for $d = 0$.