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Euler's transform and a generalized Omori's law

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Abstract

The self-replication process of the statistical events generated by an original, main event and described by a finite distribution leads to a generalized Omori distribution singular at origin. The two distributions are related to each other by Euler's transform. The self-consistency of the generating process requires an exponential law for the finite distribution, which gives rise to the original Omori's law associated to the seismic activity accompanying a major earthquake.

Singular distributions of power-law type $P(x) \sim 1/x^{\beta}$, for $x > 0$ and $\beta > 1$, seem to be ubiquituous.[1] Originally, they may have been introduced by Omori in 1894[2, 3] for describing the distribution of the seismic aftershocks with $\beta = 1^{+}$ and x denoting the time elapsed from the occurrence of the main seismic shock at $x = 0$. Such distributions, which may be called Omori-type singular distributions, are widely used in analyzing the seismic activity accompanying a major earthquake, both as aftershocks and foreshocks, as well as in a great variety of other situations.[4]-[11] The power-law bears also relevance on a critical-point theory for the accompanying seismic activity, and other similar phenomena, especially in connection with the self-organized criticality.[12]-[14] In view of their possible non-integrability, such power-laws are usually defined over a range $x_c < x < D$, as large as possible, where x_c is a lower-bound cutoff and $D \gg x_c$ is an upper-bound cutoff. The cutoff parameter x_c may be set zero (for $0 < \beta < 1$, for instance), and D may be extended to infinite (for $\beta > 1$, for instance).

It is shown here that such Omori-type singular distributions may arise from self-replicating events, originally produced by a main event according to a finite distribution. The two distributions are related to each other by Euler's transform, which provides a generalized form for Omori's law. The self-consistency of the production process imply a self-generating original distribution, which is given by an exponential law. The distribution of all the events produced in such a process, self-replication included, is the original Omori's law.

Let $p(x) = dN/dx$ be a finite distribution of events N over the range $x > 0$. The number $dN_0 = p_0 dx$ of events placed at origin, where $p(0) = p_0$, can be viewed as the number of the main events, while the rest of events, distributed over $x > 0$, can be viewed as produced by the main events at a rate $r(x)$ given by

$$p(x) = p_0 r(x) \ . \tag{1}$$

The self-replication process implies a distribution $P(x)$ obeying the relationship

$$P(x) = p(x) + r(x)P(x) = p(x) + \frac{p(x)}{p_0}P(x) \ . \tag{2}$$

It follows that the distribution $P(x)$ is given by

$$P(x) = \frac{p(x)}{1 - p(x)/p_0} \ , \tag{3}$$

which is Euler's transform between $p(x)/p_0$ and $-P(x)/p_0$. The distribution $P(x)$, as given by (3), corresponds to all the events generated in the process of producing accompanying events by the main events placed at $x = 0$. It is worth noting that $P(x)$ is singular at origin.

Distributions $P(x)$ as given by Euler's transform (3) can be considered for a general form of generating distributions $p(x)$, which amounts to including only the self-replication process for the accompanying events produced by $p(x) = p_0 r(x)$. For this general case, the series expansion $p(x) = p_0 - p_1 x ...$ can be considered in the neighbourhood of $x = 0$, leading to Omori's law $P(x) = p_0 x_0/x$ for $x \ll x_0 = p_0/p_1$. Euler's transform (3) provides a general representation $P(x) = p_0/h(x)$ for such singular distributions, where $h(0) = 0$ and $h(\infty) \rightarrow \infty$ (such that, preferrably, $P(x)$ is integrable at infinite). It implies $p(x) = p_0/(1 + h) \simeq p_0(1 - h)$ for $x \rightarrow 0$. Such a representation may be regarded as a generalized Omori-type distribution. For $h(x) \sim x^\beta$, $\beta > 0$, power-law distributions $P(x) \sim 1/x^\beta$ are obtained (an upper-bound cutoff D is necessary for $0 < \beta \leq 1$, as well as a lower-bound cutoff x_c for $1 \leq \beta$).

Since the accompanying events are produced by the main events at a rate $r(x) = p(x)/p_0$, and since the events are not differentiated otherwise except by their position x, it follows that the distribution p may also be produced at $x + y$ by its value at x multiplied by rate $r(y)$, *i.e.*

$$p(x + y) = p(x)r(y) \ , \tag{4}$$

for any $x, y > 0$. This distribution may be viewed as a self-generating distribution, and equation (4) expresses a self-consistency character of the distribution $p(x)$. Equation (4) can also be written as $p(x + \Delta x) = r(\Delta x)p(x)$, or $dp/dx = (-p_1/p_0)p(x)$, where $-p_1 = p'(0) < 0$ is the first derivative of $p(x)$ at origin. It follows immediately, from (1) and (4), that distribution $p(x)$ is given by an exponential law, $p(x) = p_0 e^{-p_1 x/p_0}$. It can be transformed into a normalized probability distribution $p(x) = p_0 e^{-p_0 x}$.

Inserting the exponential distribution $p(x) = p_0 e^{-p_0 x}$ in (3) the distribution

$$P(x) = \frac{p_0}{e^{p_0 x} - 1} \ , \tag{5}$$

is obtained, which is Omori's law $P(x) = 1/x$ for $p_0 x \ll 1$. It is customary to introduce a lower-bound cutoff x_c and to extend $1/x$ to infinite as $x_c^{\beta-1}/x^\beta$, where $\beta = 1^+$, such that

$$\int_{x_c}^{\infty} dx \frac{p_0}{e^{p_0 x} - 1} = \int_{x_c}^{\infty} dx (x_c^{\beta-1}/x^\beta) \ . \tag{6}$$

Equation (6) gives the exponent $\beta = 1 - 1/\ln(p_0 x_c) = 1^+$ in the limit $x_c \rightarrow 0$.

It might be noted that $P(x)$ as given by (5) is, formally, a Bose-Einstein-type occupation number (in two dimensions) for an exponential, Boltzmann-type, distribution $p(x)$. The self-replication equation (2), which describes a geometric series, has also a formal resemblance to Dyson's equation in the theory of interacting many-body ensembles. Equation (5) can also be viewed as a generalized Omori's law.

In conclusion, it may be said that self-replication processes at a rate $r(x) = p(x)/p_0$ for a generating distribution $p(x)$ of events accompanying the main events placed at $x = 0$ lead to Omori-type singular distributions as given by Euler's transform (3). Such distributions include power-type distributions of the form $1/x^\beta$, where $\beta > 0$. The self-consistency of the generating process requires a self-generating distribution $p(x)$, which is given by an exponential law, and which leads to the original Omori's law $1/x^\beta$, with $\beta = 1^+$.

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