Journal of Theoretical Physics

Founded and Edited by M. Apostol **108** (2005)

ISSN 1453-4428

A model of seismic focus and related statistical distributions of earthquakes

B.-F. Apostol
Department of Seismology, Institute of Earth's Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apostol@infp.infp.ro

Abstract

A growth model for accumulating seismic energy in a localized seismic focus is described, which introduces a fractional parameter r on geometrical grounds. The model is employed for deriving a power-type law for the statistical distribution in energy, where the parameter r contributes to the exponent, as well as corresponding time and magnitude distributions for seismic events. The magnitude distribution is applied to Vrancea earthquakes, in order to assess relevant statistical parameters for this seismic region. The accompanying seismic activity of foreshocks and aftershocks is also discussed in connection to this approach, the associated Omori distributions are derived, and the time dependence of the magnitudes and the rate of released energy are given. It is shown that Omori's distribution arises from a self-replication process of a generating distribution, the two distributions being inter-related by Euler's transform. The self-consistency of the process generating the accompanying seismic events requires an exponential law for the generating distribution. A generalization of Omori's law is also included.

PACS: 91.30.Bi, 91.30.Dk, 91.30.-f, 91.35.Cb

Keywords: Seismic source; Recurrence time; Statistical distributions; Omori's law; Gutenberg-Richter law

1 Introduction

The physical mechanisms of the seismic sources are largely unknown, and the patterns exhibited by earthquakes in space and time are still a matter of debate (see, for instance, Bak et al., 2002; Sornette, 2003). The present paper introduces a model of accumulating seismic energy in a localized critical focal zone, and derives statistical distributions of earthquakes in time, energy and magnitude, which seem to enjoy a certain consensus, at least partially. The focus model includes a fractional parameter r, derived on geometrical grounds, which turns out to be an Omori-type parameter in the power-law distribution of earthquakes with respect to energy. It affects also the distribution in magnitude, the recurrence law and the mean recurrence time of the earthquakes. The magnitude distribution is employed for analyzing a set of 1999 earthquakes with magnitude greater than $M = 3$, which occurred in Vrancea, Romania, between 1974 and

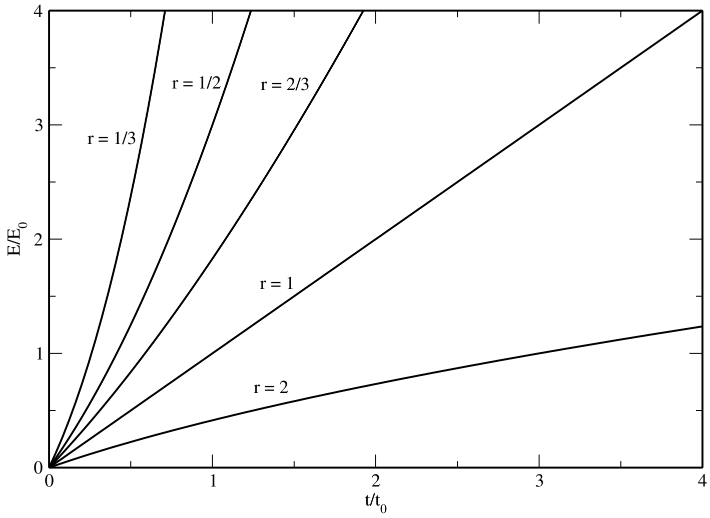

Figure 1: Reduced energy *vs* reduced time, as given by equation (6) for various values of the parameter r

2004, in order to assess statistical parameters which are relevant for this seismic zone. The associated regime of seismic activity in the neighbourhood of a main, "regular" sesimic shock is also discussed, the corresponding Omori distributions in time, magnitude and energy are derived, and the time dependence of magnitudes and the rate of released energy are given. It is shown that Omori's law originates in a self-replication process for a generating distribution of accompanying seismic events, and the self-consistency of the generating process requires an exponential law for the generating distribution. The two distributions are inter-related by Euler's transform, which provides a generalization of Omori's law.

It is widely agreed that the seismic energy E released in an earthquake is related to the earthquake's magnitude M by the Gutenberg-Richter-type relationship (Gutenberg and Richter, 1944, 1954; Kanamori and Anderson, 1975)

$$\ln E = a + bM \ . \tag{1}$$

Statistical analysis of moderate and strong earthquakes ($5.8 < M < 7$), which are probably most prone to represent a statistical ensemble, indicates values $a \simeq 10$ and $b \simeq 3.5$ (in decimal logarithms $a \simeq 4.4$ and $b \simeq 1.5$) for energy measured in J (Bullen, 1963). (The error in seismic energy may be up to a factor of 10). These numerical values may be adopted for the present purpose,[1] although the considerations made herein do not depend critically on such numerical values. Parameter a in (1) indicates the existence of a threshold energy $E_0 = e^a$ ($E_0 \simeq 4.4 \cdot 10^4$J), so that equation (1) can be recast as $E/E_0 = e^{bM}$.

It is customary to assign a region of characteristic length R to the seismic energy E, through $E \sim R^3$, and, similarly, a characteristic threshold length R_0 can be associated to the threshold energy $E_0 \sim R_0^3$, leading to

$$\ln(R/R_0) = \beta M \ , \tag{2}$$

where $\beta = b/3 = 1.17$. The two characteristic lengths R and R_0 have a double meaning, at least: on one side, they may be associated with the central core of the critical focal zone where the

[1] There are various representations for energy E and magnitude M in the Gutenberg-Richter relationship (1), as depending on various practical conventions, the most used being related to the seismic moment. All of them obey a general relationship of the form given by (1), and their specific definitions are immaterial for the present purpose .

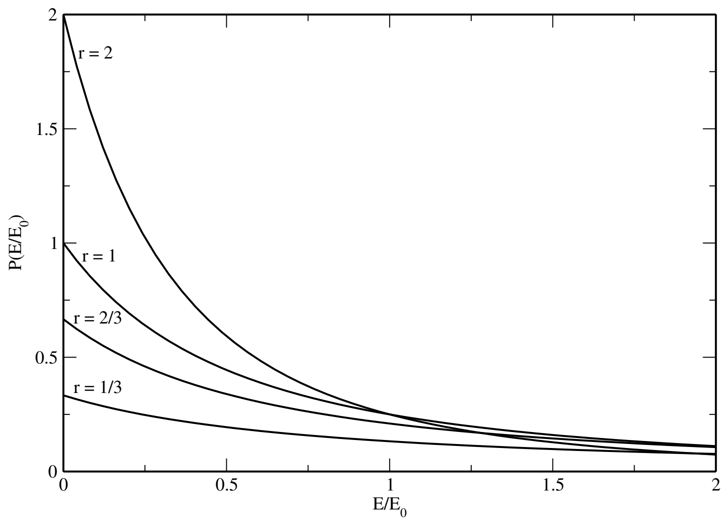

Figure 2: Energy probability distribution *vs* reduced energy, as given by equation (9), for various values of the parameter r

seismic energy accumulates, and, on the other, R may correspond to the characteristic length of the seismic region disrupted by the earthquake, R_0 being in this case a scale length. The empirical evidence in the latter case seems to support an equation of the type (2) (Buffe and Varnes, 1993; Bowman et al., 1998, and references therein).

2 A model of seismic focus

It is assumed that the characteristic lengths R and R_0correspond to a localized critical focal region where the seismic energy builds up by mechanical tension. It is also reasonable to assume that the process of accumulating energy in the seismic focus exhibits a uniform velocity $\mathbf{v}$, so that the accumulation of the seismic energy in focus obeys the continuity equation

$$\partial E/\partial t = -\mathbf{v} gradE \ , \tag{3}$$

where t denotes the accumulation time. Further on, the same value v of the velocity may be assumed along all three spatial coordinates, and the spatial variation of energy along each coordinate is represented as $(E+E_0)/(R+R_0)$. By such assumptions equation (3) becomes

$$dE/dt = (1/r)v\frac{E+E_0}{R+R_0} \ , \tag{4}$$

where $r = 1/3$. The factor $1/r = 3$ in front of (4) arises, therefore, from pure geometric reasons. Since for other, more special, geometries of the critical focal zone this factor may differ from 3, notation r is preferred in the interest of the generality of the treatment. For the present purpose the value of this parameter is taken as $r = 1/3$. Equation (4) leads also to consider the accumulation time $t = R/v$ as well as the threshold time $t_0 = R_0/v$, so it becomes

$$dE/dt = (1/r)\frac{E+E_0}{t+t_0} \ . \tag{5}$$

The solution of (5) is obtained straightforwardly as

$$1 + t/t_0 = (1+E/E_0)^r \ . \tag{6}$$

For large values of energy E ($E \gg E_0$) solution (6) reads $t/t_0 \simeq (E/E_0)^r = R/R_0$, or

$$t \simeq t_0(E/E_0)^r = t_0 e^{\beta M} \ , \tag{7}$$

where the Gutenberg-Richter law (1) is used and $\beta = br = b/3 = 1.17$. Equations (6) and (7) are the basic equations of the present model of seismic focus. According to equation (5), such a model looks like a growth model, with a typical power-law as given by (6). The reduced energy E/E_0 is shown in Fig.1 as function of the reduced time t/t_0 as given by (6), for several values of the parameter r.

3 Statistical distributions

Let N_0 be the number of earthquakes during a long time T, characterized by the average threshold time $t_0 = T/N_0$, where N_0 is very large. The cutoff parameter t_0 may be viewed as the seismicity rate. Similarly, the frequency of N earthquakes characterized by time t can be written as $N/N_0 = 1/(1+t/t_0)$.[2] Hence, it follows straightforwardly the temporal probability distribution

$$P(t)dt = -d(\frac{1}{1+t/t_0}) = \frac{1}{(1+t/t_0)^2}dt/t_0 \ , \tag{8}$$

or, making use of (6), the probability distribution in energy

$$P(E)dE = \frac{r}{(1+E/E_0)^{1+r}}dE/E_0 \ . \tag{9}$$

Similar power-law distributions in energy have been derived recently by employing Tsalis entropy for the fragmentation of a dynamical fault-planes model (Sotolongo-Costa and Posada, 2004). Such distributions are sometimes called Omori-type distributions, where r is an Omori parameter. The energy probability given by (9) is shown in Fig.2 as function of the reduced energy E/E_0 for various values of the parameter r.

Making use of the energy distribution (9) and the Gutenberg-Richter law (1) the magnitude distribution

$$P(M)dM = \beta e^{-\beta M}dM \tag{10}$$

is obtained straightforwardly, for large energies $E \gg E_0$. The number ΔN of seisms with magnitude between M and $M + \Delta M$ is given by $\Delta N/N_0\Delta M = P(M)$, or

$$\lg(\Delta N/T) = A - BM \ , \tag{11}$$

where $A = \lg(\beta\Delta M/t_0)$ and $B = \beta/2.3$. Such a linear relationship has been checked for a large amount of earthquakes, and $A \simeq 4.6$ and $B \simeq 0.6$ were obtained, for instance, for $5.8 < M < 7.3$ (and $\Delta M = 0.1$) (Bullen, 1963). These values may be adopted here for the present purpose, though the numerical values of such quantities do not affect the results presented herein. Making use of the value for the parameter$B$, it is obtained $\beta \simeq 1.38$, in fair agreement with the value $\beta = 1.17$ given here. Similarly, a global rate of seismicity $1/t_0 \sim 10^{5.5}$ per year is obtained from the value of the parameter A, which is consistent with estimations of cca $10^5 - 10^6$earthquakes per year, on average (Bullen, 1963). There are appreciable deviations from the Gutenberg-Richter

[2] The total number of earthquakes may also be taken as $\mathcal{N} \simeq (T/t_0)\ln(T/t_0)$, which amounts to renormalizing the threshold time to $\tilde{t}_0 = t_0/\ln(T/t_0) \to 0$ for $T \to \infty$. A similar renormalization holds also for the threshold energy E_0, the results being thereby free from arbitrary cutoff parameters.

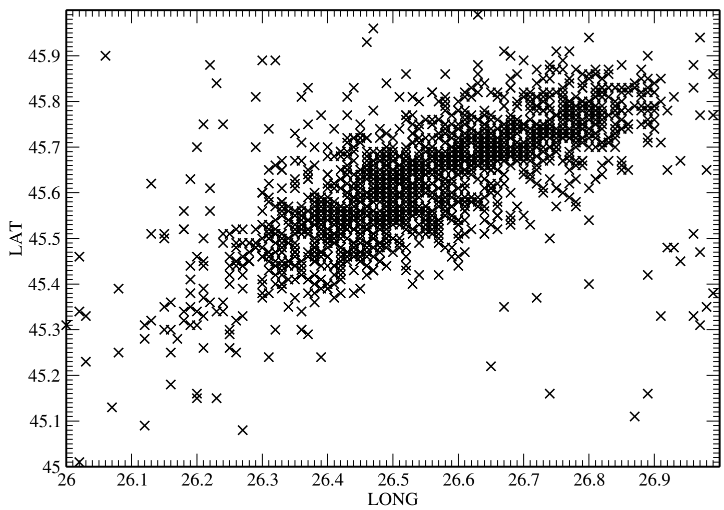

Figure 3: Geographical distribution in longitude and latitude of Vrancea earthquakes between 1974-2004 (magnitude $M > 3$, Romanian Earthquake Catalogue, 2005)

linear relationship (11) for extreme values of the magnitude.[3] For low values of M such deviations are consistent with the exact relationship $P(M) = be^{bM}/(1+e^{bM})^{1+r}$ derived from the distribution given by (9) and the Gutenberg-Richter law, but for large values of the magnitude these deviations may indicate either that large seismic events are not statistical events, or the deviations may be ascribed to a magnitude saturation phenomenon.

It is also convenient to introduce the so-called recurrence law, or the excedence rate, which gives the number $N_{ex}$of earthquakes with magnitude greater than M. The corresponding probability is readily obtained from (10) as $P_{ex} = e^{-\beta M}$, so the excedence rate reads

$$\ln(N_{ex}/T) = -\ln t_0 - \beta M \; . \tag{12}$$

This relationship is currently employed for analyzing the earthquake statistical distributions in magnitude. A recent analysis seems to indicate a certain universality in the value of the β slope ($B = \beta/2.3 \simeq 0.6$) (Kagan, 1999; Rabinovitch et al., 2001).

It is worth noting that equation (7) may be viewed as providing the mean recurrence time $t_r = t_0 e^{\beta M}$ for the occurrence of earthquakes of magnitude M (energy $E \gg E_0$). In fact, the mean recurrence time of earthquakes with magnitude in the range M to $M+\Delta M$ is of interest. According to (10) the rate of such earthquakes is given by $\Delta N/T = (\beta \Delta M/t_0)e^{-\beta M}$, so the mean recurrence time can be obtained as

$$t_r = (t_0/\beta \Delta M)e^{\beta M} \; . \tag{13}$$

If the seismicity rate t_0 is known, this equation may be used to predict the mean recurrence times. However, it must be noted that the accuracy of such predictions is, in fact, very poor. Indeed, imposing a mean recurrence time t_r, the temporal distribution $(1/t_r)e^{-t/t_r}$ is obtained immediately from the maximum of the entropy, for instance. The deviation in the recurrence time defined as $(\bar{t^2})^{1/2} - \bar{t}$ is $(\sqrt{2}-1)t_r$ for such distributions, which amounts to cca 41% of the mean recurrence time t_r. It is a very large deviation to be of practical use.

[3]For instance, parameter B in (11) may double its value, becoming $B \sim 1$, for very strong earthquakes

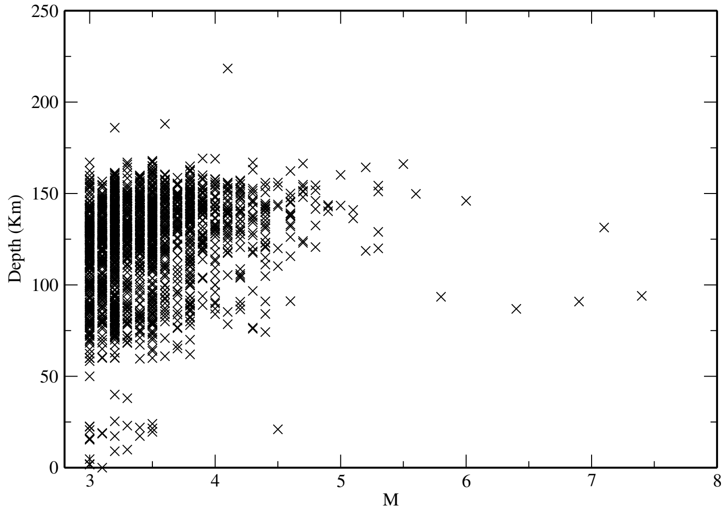

Figure 4: Depth distribution of Vrancea earthquakes between 1974-2004 (magnitude $M > 3$, Romanian Earthquake Catalogue, 2005)

M=3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4.0	4.1
ΔN=245	230	362	143	176	230	109	72	147	56	48	41

M=4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	5.0	5.1	5.2	5.3	5.4
ΔN=33	23	27	7	15	8	6	3	2	2	2	4	0

M=5.5	5.6	5.7	5.8	5.9	6.0	6.1	6.2	6.3	6.4	6.5	6.6	6.7
ΔN=1	1	0	1	0	1	0	0	0	1	0	0	0

M=6.8	6.9	-	7.1	-	7.4
ΔN=0	1	-	1	-	1

Table 1: Magnitude distribution of Vrancea earthquakes from 1974 to 2004 (magnitude $M > 3$, Romanian Earthquake Catalogue, 2005)

4 Statistical analysis of Vrancea earthquakes

In order to illustrate the statistical distributions given above, as well as to derive statistical parameters of interest, the magnitude distributions (10)- (12) are applied to the seismic region Vrancea, Romania. The seismic focal zone in Vrancea is located approximately at 45.7°N latitude and 26.6°E longitude. This focal zone is the source of a noteworthy seismic activity, ranging in depth from 80km to 150km, with earthquakes greater than $M = 7$ in (moment) magnitude sometime (for instance, $M = 7.4$, March 4, 1977, depth 94km, or $M = 7.1$, August 30, 1986, depth 131km). A set of 1999 data is used, comprising Vrancea earthquakes in period 1974-2004 with magnitude greater than $M = 3$, as given in Romanian Earthquake Catalogue (2005). The geographical distribution in latitude and longitude of Vrancea earthquakes with magnitude $M > 3$ from 1974 to 2004 is shown in Fig.3, while de depth distribution of the same set of data is shown in Fig.4. The distribution $\Delta N(M)$ of these data is given in Table 1, for $\Delta M = 0.1$ (magnitude

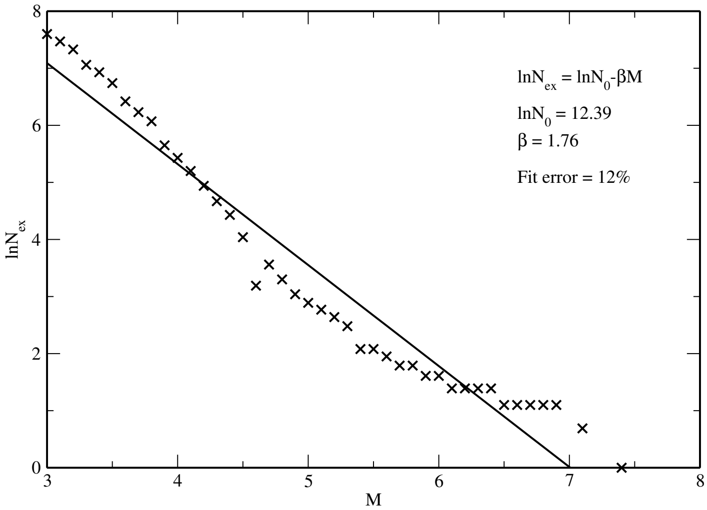

Figure 5: Recurrence law $\ln N_{ex} = \ln N_0 - \beta M$ fitted to cumulative distribution of data given in Table 1.

error 0.1).[4]

The recurrence law (12) is employed in the form

$$\ln N_{ex} = \ln N_0 - \beta M \tag{14}$$

to fit the cumulative data, as shown in Fig.5. The fit gives $\ln N_0 = 12.39$ and $\beta = 1.76$, with 12% accuracy. It corresponds to a rate of seismicity $-\ln t_0 \simeq 8.99$ (for 30 years) and to coefficient $B = \beta/2.3 \simeq 0.76$. The logarithmic distribution (11) is used in the form

$$\ln \Delta N = \ln C - \beta M \tag{15}$$

for a similar fit, where $C = \beta N_0 \Delta M$, $\ln C = 13.19$, $\beta = 2.36$ with $\sim 10\%$ accuracy ($\Delta M = 0.1$), as shown in Fig.6. It follows a seismicity rate $-\ln t_0 \simeq 11.23$ and a coefficient $B = \beta/2.3 \simeq 1.03$. A third fit is reported by using the magnitude distribution $\Delta(M) = C \exp(-\beta M)$ as given by (10) directly for the data $\Delta N(M)$ given in Table 1. It is shown in Fig.7, the parameters being $\ln C = 10.35$ and $\beta = 1.54$ (accuracy 32%). It corresponds to a seismicity rate $-\ln t_0 \simeq 8.82$ and a coefficient $B = \beta/2.3 \simeq 0.67$.

The three fits shown in Figs.5-7 are of different quality and accuracy. Their results must be taken with great caution. The direct exponential fit to the data is probably of the best quality, because it takes into account all the data, in contrast to the logarithmic fits were vanishing data are excluded. Unfortunately, its accuracy is poor, due to the abrupt variation of the exponential function and data scattering. Similarly, the best accuracy (10%) is for the logarithmic fit given by equation (15), as the function variation is the smallest. However, the quality of this fit is poor, as it loses the greatest number of data (all vanishing data). Finally, the recurrence law (14) produces a fit which is situated in the middle of such an estimation, of moderate accuracy and quality. Under these circumstances, it is reasonable to use an average value for the seismicity rate $-\overline{\ln t_0} = (8.99 + 11.23 + 8.82)/3 = 9.68$ and an average $\bar{\beta} = (1.76 + 2.36 + 1.54)/3 = 1.89$

[4]The data employed here, as collected from the Romanian Earthquake Catalogue (2005), correspond to location 45°N to 46°N latitude and 26°E to 27°E longitude. A few surface, or middle depth, earthquakes reported for Vrancea region are also included in the data set. Magnitude M_w in Romanian Earthquake Catalogue (2005) is taken here for the moment magnitude denoted by M.

(accuracy $\sim 18\%$). It is worth noting that a close value $\bar{\beta} = 2.1$ is obtained by using directly the mean $\bar{\beta}\Delta M = \ln(\overline{\Delta N_i / \Delta N_{i+1}})$ for data in Table 1. The value $\beta = 1.89$, which corresponds to a coefficient $B = \beta/2.3 = 0.82$, indicates a value $r = \beta/b = 0.54$ for the parameter r of the focus model, where $b = 3.5$ is used. This may show that the geometry of the Vrancea focus is different from a point-like source accumulating seismic energy with a uniform velocity, resembling more a two-dimensional geometry ($1/r = 1.85$), probably with slightly different accumulating velocities. Such a conclusion can be corroborated with the spatial distributions shown in Fig.3 and Fig.4. Making use of the values obtained here for the average seismicity rate and average β-parameter, one may attempt to estimate the mean recurrence time, by using equation (7). The value $t_r \simeq 34.9$years is obtained this way, for the mean reacurrence time of earthquakes with magnitude $M > 7$ in Vrancea region. It must be recalled in this context that the error of such an estimation is $\sim 41\%$, *i.e.* ~ 14.3years (leaving aside the errors in determining the statistical parameters t_0 and β). This periodicity can be checked against data in Fig.8, where Vrancea earthquakes with (moment) magnitude $M > 6$ are shown for the last two hundreds years (Romanian Earthquake Catalogue, 2005).

Finally, it may be worth noting the oscillations in $\ln N_{ex}$ in Fig.5, as well as similar oscillations in $\ln \Delta N$ in Fig.6, or in Fig.7 (though of different periodicity and poorer quality), whose origin is still debated. Such oscillations may be associated, for instance, with the logarithmic oscillations in the critical-point theory distributions (Sornette, 1998; 2003).

5 Accompanying seismic activity. Omori's law

The above description may be viewed as pertaining to "regular" earthquakes, characterized by a mean recurrence time. Similarly, the energy associated to such times, as given by (6) or (7), may be viewed as a mean energy. Such "regular" seismic events may be accompanied by an associated seismic activity, like foreshocks and aftershocks, in which case a "regular" earthquake is referred to as the main shock. Since 1894, when Omori suggested that seismic aftershocks are distributed according to $\sim 1/\tau^\gamma$, where $\gamma = 1^+$ and τ denotes the time elapsed from the main shock (Omori, 1894), the seismic activity accompanying a major earthquake is a matter of debate. One of the major difficulties in advancing knowledge in this subject is the lack of means for distinguishing between seismic events genuinely accompanying a main shock and other, "regular" seisms, superposed over the associated seismic activity, which may possibly belong to other "regular" time series of seismic activity, without any relationship with the main seismic shock. Statistical distributions of such events, both in time, magnitude and energy, may help in operating such a distinction, and it was precisely in this direction where progress has been recorded recently, especially in connnection with the critical-point theory of foreshocks and aftershocks, as based on self-organized criticality (Sornette at al., 1992; Sornette, 1998).

It is assumed here that there may exist an associated seismic activity accompanying a main seismic event, as seismic foreshocks and aftershocks, and this whole "secondary" seismic activity forms a statistical ensemble, *i.e.* is described by probability distributions.

Let the main shock occurs at a critical time $t_c = 0$, and measure time τ of the accompanying seismic activity with respect to this initial moment of time. Time τ takes both positive values, for aftershocks, and negative values, for foreshocks. As this seismic activity corresponds to pairs of events separated by time τ, then the corresponding statistical distributions are functions of the absolute value $|\tau|$ of time τ, as pointed out in earlier studies (for instance, Papazachos, 1975). It is shown in **Appendix** that the associated seismic activity proceeds by the self-replication of

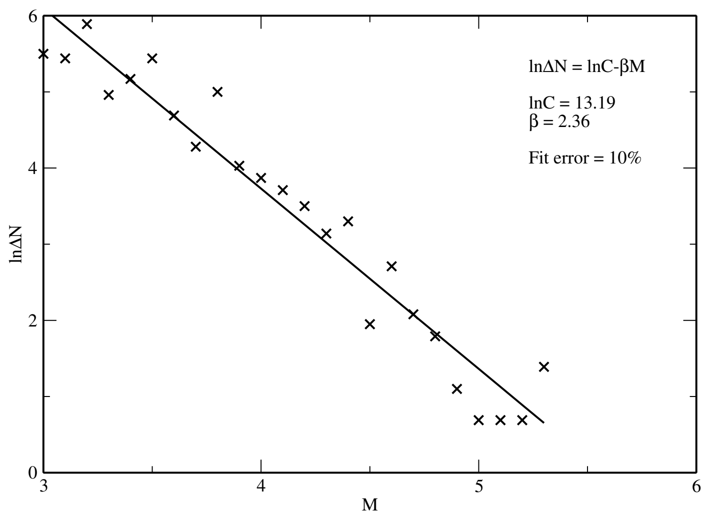

Figure 6: Logarithmic distribution $\ln \Delta N = \ln C - \beta M$ fitted to data given in Table 1.

a generating distribution of accompanying events, the self-consistency of the process requiring an exponential form for the generating distribution. It amounts to viewing the accompanying seismic activity as a relaxation to equilibrium of the seismic zone, and the corresponding statistical distribution $p(\tau)$ can be obtained formally by using the principle of the maximal entropy $S = -\int d\tau \cdot p(\tau) \ln p(\tau)$. In order to fully characterize this associated seismic activity, a mean value t_c' of its duration may be introduced, where t_c' may be viewed as a characteristic scale time. By standard procedure the temporal probability distribution

$$p(\tau) = \alpha e^{-\alpha|\tau|} \ , \ \ \alpha = 1/t_c' \tag{16}$$

is obtained straightforwardly, as the generating distribution for seisms accompanying a main shock. In general, the characteristic time t_c' may depend not only on the nature of the seismic source and the seismic zone, but also on the magnitude of the main shock. On the other hand, the distribution of the accompanying events can be obtained directly from (8) by expanding the temporal probability of the main shocks in powers of $|\tau|$ in the neighbourhood of a main shock with mean recurrence time t_r. It is easy to see that replacing $t = t_r$ by $t = t_r + |\tau|$ in (8), where $|\tau| \ll t_r$, the time distribution $p(\tau) \sim (1 + |\tau|/t_r)^{-2} \sim e^{-2|\tau|/t_r}$ can be extracted from the pair distribution, as corresponding to the accompanying seismic activity. It follows that parameter α in (16) is given by $\alpha = 2/t_r$, and the characteristic time $t_c' = t_r/2$, where t_r is the mean recurence time of the main shock, as given by (7) or (13). For large values of time t_r the distribution of the accompanying events has a long tail, but the corresponding time probability is very low. In contrast, the acompanying seismic activity ends quickly for small main shocks, characterized by a small value of the mean recurrence time t_r.

It is shown in **Appendix** that the self-replication process of the generating distribution given by (16) leads to the distribution $P(\tau) = \alpha/(e^{\alpha|\tau|} - 1)$ for the seismic events accompanying a major earthquake, which is Omori's law $P(\tau) = 1/|\tau|$ for $\alpha\tau \ll 1$. It may be extended to $\tau \to \infty$ as $P(\tau) = \tau_c^{\gamma-1}/|\tau|^\gamma$, where $\gamma = 1^+$ and τ_c is a lower-bound cutoff. This result is valid in general, for any finite generating distribution p, the two distribution p and P being inter-related by Euler's transform. This relationship provides also a generalized Omori's law, which is included in **Appendix**. According to Omori's law, the accompanying events are concentrated in the neighbourhood of the lower-bound cutoff τ_c. It might also be noted, according to Omori's law, that number n of associated seismic events goes like $dn/d\tau \sim 1/|\tau|$ (Utsu, 1961; Sornette et

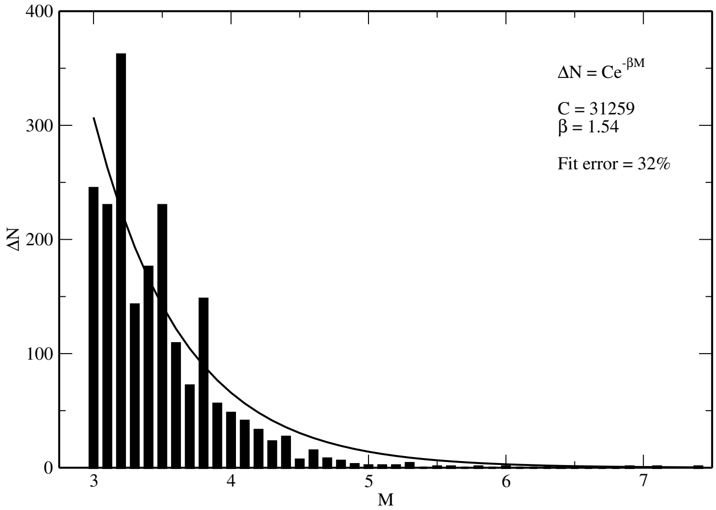

Figure 7: Exponential distribution $\Delta N = C\exp(-\beta M)$ fitted to data in Table 1.

al., 1992). An exponential distribution $p(\tau)$ of the form given by (16) is shown in Fig.9, together with its corresponding Euler's distribution $P(\tau)$. For comparison, it is also shown in Fig.9 the normalized distribution $P(\tau)$ as well as the normalized Omori's law $(\gamma-1)\tau_c^{\gamma-1}/\left|\tau\right|^{\gamma}$. These distributions illustrate the singular character of the self-replication process at origin, their abrupt decrease when normalized, together with a possible long tail.

A distribution similar to (16) holds also for the difference in magnitude of the associated seisms with respect to the main shock. Indeed, according to (10), the magnitude distribution can be written as $\sim e^{-\beta m}e^{-\beta M}$ for a main shock of magnitude M_0, where $m = M_0 - M$ is the difference in magnitude between the main shock and an accompanying seismic event of magnitude M. Negative values for the statistical variable $m = M - M_0$ must be allowed in such a distribution, which leads to $\beta e^{-\beta|m|}$ for the distribution in magnitude difference, as suggested previously (Vere-Jones, 1969). It may also be noted that such a distribution can be obtained by the principle of the maximal entropy as $\beta' e^{-\beta'|m|}$, and, since this probability is equal to the probability of the main shock at $m = 0$, it follows that $\beta' = \beta$. Another observation might also be that associated seisms do follow the same exponential distribution in magnitude like the "regular" earthquakes.

It is worth noting that, by making use of the exponential distribution in magnitude difference and the temporal distribution given by (16), the time dependence $|m| = (\alpha/\beta)\left|\tau\right|$ is obtained, or $dm/d\tau = \alpha/\beta$, or, equivalently, the time dependence $M = M_0 - (\alpha/\beta)\left|\tau\right|$ of the magnitude of the accompanying seisms. It may be estimated that the associated seismic activity is extinct in time $\tau_0 = \beta M_0/\alpha = \beta M_0 t_c'$, though the long-tail values of the probability distributions of the accompanying seismic activity are very small. As described above, for small values of m ($|m| < 1/\beta$) the distribution in magnitude difference obeys the same Omori-type law $\sim dm/\left|m\right|$ (the lower bound correponding to $m_c = (\alpha/\beta)\tau_c$). The mean difference in magnitude $\bar{m}$ vanishes for the distribution $\beta e^{-\beta|m|}$ ($\bar{m} = 0$), so it is reasonable to employ the dispersion $\delta m = \overline{(m^2)}^{1/2} = \sqrt{2}/\beta$ as a measure of the average deviation in magnitudes of the accompanying seismic activity. Such an estimation is also consistent with the assumption that the associated seismic activity represents a relaxation regime of the seismic activity. Making use of $\beta \simeq 1.17$ the value $\delta m = \sqrt{2}/\beta \simeq 1.2$ is obtained, which is suggestive for the numerical value indicated by Bath's empirical law (Bath, 1965). A similar analysis, though on a different conceptual basis, was made recently for the accompanying seismic activity (Helmstetter and Sornette, 2002; Console et al., 2003; Helmstetter et al., 2004). It might be noted that the self-replication process is not included in estimating the

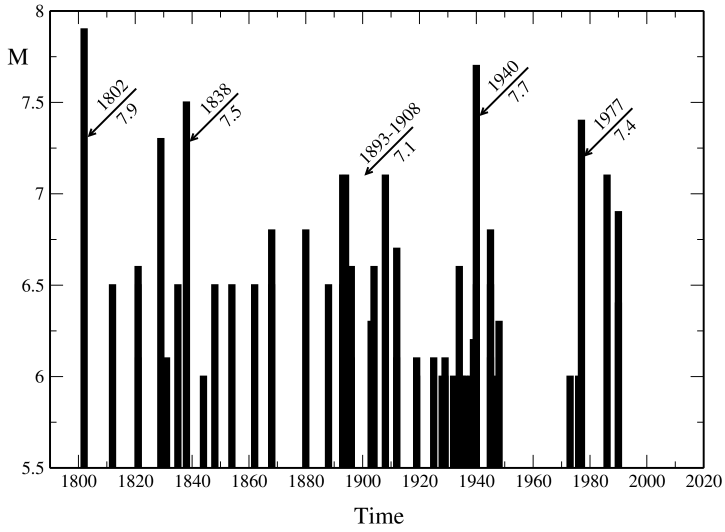

Figure 8: Vrancea earthquakes with (moment) magnitude $M > 6$ in the last two centuries (Romanian Earthquakes Catalogue, 2005)

magnitude dispersion, and the variance $\delta m = \sqrt{2}/\beta$ occurs in time $\tau_B = (\beta/\alpha)\delta m = \sqrt{2}t'_c$.

The energy distribution given by (9) can also be written as $P(E) = (rE_0^r/E^{1+r})(1 + E_0/E)^{-1-r}$, where the factor in the first paranthesis may be assigned to energy E_{max} of the main seismic shock, while the factor in the second paranthesis may be assigned to energy $\varepsilon = E$ corresponding to an accompanying seism. Such an approximation is valid for values of the energy E close to the energy E_{max}, and serves to disentangle the accompanying seismic activity from the main shock. It is consistent with the afore-reached conclusion that the associated seismic activity is governed by the same distribution in magnitudes as the main activity. The resulting decomposition indicates that the statistical variable corresponding to energy for the accompanying seisms is actually $x = 1/\varepsilon$, so that the "energy" distribution $p(x) \sim (1 + E_0/\varepsilon)^{-1-r} = \exp[-(1 + r)\ln(1 + E_0/\varepsilon)]$ can be written down for the associated seismic activity, or

$$p(x) \simeq E_o(1 + r)e^{-(1+r)E_0 x} \quad , \quad x = 1/\varepsilon \ . \tag{17}$$

It may be noted that this distribution is similar to the exponential distributions in time, or magnitude, with a characteristic scale energy $(1+r)E_0$. By comparing (17) and (16) the time dependence $\varepsilon = (1 + r)E_0 t'_c / |\tau|$ of the released energy is obtained straightforwardly, which corresponds to the rate

$$d\varepsilon/d\,|\tau| = -(1 + r)E_0 t'_c/\tau^2 \tag{18}$$

of the energy released in the accompanying seismic activity. Such an $\sim 1/\tau^2$- law seems to be supported by empirical data (Utsu, 1961; Sornette et al., 1992). Similarly, the magnitude dependence $\varepsilon = (1 + r)E_0/\beta\,|m|$ is obtained for the released energy, as well as an Omori-type law $\sim dx/x \sim -d\varepsilon/\varepsilon$ for small values of x (large values of released energy ε).

In conclusion, it may be said that a model is introduced here for the accumulation of the seismic energy in a localized focus, which implies a geometric parameter r. Statistical distributions in time, energy and magnitude are derived on this basis for regular earthquakes, and corresponding Omori's distributions are also derived for the seismic activity accompanying a main seismic shock. Time dependence (18) of the released energy in an accompanying seismic activity is given. It is also shown that Omori's law implies a self-replication process for a generating distribution of accompanying seismic events, which is given by an exponential law. The two distributions are

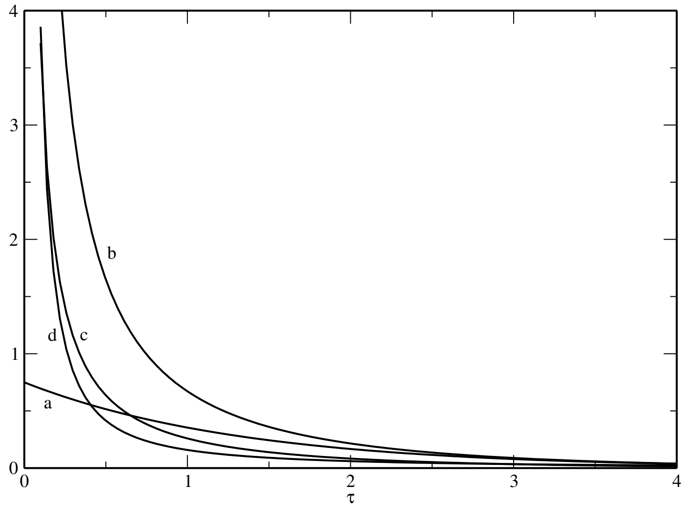

Figure 9: Exponential distribution $\alpha\exp(-\alpha\tau)$ for $\alpha = 0.75$ (curve a), self-replication distribution $\alpha/[(\exp(\alpha\tau) - 1]$ for $\tau > \tau_c = 1$ (curve b), normalized self-replication distribution (curve c) and normalized Omori's law $(\gamma-1)\tau_c^{\gamma-1}/\tau^\gamma$ for $\gamma = 1 - 1/\ln(\alpha\tau_c)$ (curve d). The normalized distributions exhibit a sudden fall, and a possible long tail for small α.

inter-related by Euler's transform, which provides also a generalized Omori's law. It is also shown how to employ such theoretical considerations for analyzing the seismic activity, by making use of the seismic data of Vrancea region.

Acknowledgments

The author is indebted to the members of the Institute of Earth's Physics, Magurele-Bucharest for enlightening discussions and to dr. L. C. Cune for help with graphics and statistical analysis.

Appendix

Generalized Omori's law and Euler's transform

Let $p(x) = dn/dx$ be a finite distribution over the range $x > 0$. The number $dn_0 = p_0dx$ of events placed at origin, where $p(0) = p_0$, can be viewed as the number of main events, while the rest of events, distributed over $x > 0$, can be viewed as being produced by the main events at a rate $r(x)$, such that

$$p(x) = p_0r(x) \ . \tag{19}$$

Since the events are not differentiated otherwise except by their position x, it follows that distribution p is also produced at $x + y$ by its value at x multiplied by rate $r(y)$, *i.e.*

$$p(x+y) = p(x)r(y) \ , \tag{20}$$

for any $x, y > 0$. This is a self-generating distribution, and equation (20) expresses a self-consistency character of distribution $p(x)$. It may also be written as $p(x + \Delta x) = r(\Delta x)p(x)$, which leads to $dp/dx = (-p_1/p_0)p(x)$, where $-p_1 = p'(0) < 0$ is the first derivative of $p(x)$ at origin. It follows immediately, from (19) and (20), that distribution $p(x)$ is given by an exponential law $p(x) = p_0 e^{-p_1 x/p_0}$, which can be transformed into a normalized probability distribution $p(x) = p_0 e^{-p_0 x}$.

The self-replication process implies a distribution $P(x)$, giving the total number of events $P(x)dx$ in the range x to $x + dx$, which obeys the relationship

$$P(x) = p(x) + r(x)P(x) = p(x) + \frac{p(x)}{p_0} P(x) \ . \qquad (21)$$

It follows that the distribution $P(x)$ is given by

$$P(x) = \frac{p(x)}{1 - p(x)/p_0} \ , \qquad (22)$$

which is Euler's transform between $p(x)/p_0$ and $-P(x)/p_0$. The distribution $P(x)$ as given by (22) corresponds to all the events generated in the process of producing accompanying events by the main events placed at $x = 0$. It is worth noting that $P(x)$ is singular at origin. Introducing the exponential distribution $p(x) = p_0 e^{-p_0 x}$ in (22) the distribution

$$P(x) = \frac{p_0}{e^{p_0 x} - 1} \ , \qquad (23)$$

is obtained, which is Omori's law $P(x) = 1/x$ for $p_0 x \ll 1$. It is customary to introduce a lower-bound cutoff x_c and to extend $1/x$ to infinite as $x_c^{\gamma-1}/x^{\gamma}$, where $\gamma = 1^+$, such that

$$\int_{x_c}^{\infty} dx \frac{p_0}{e^{p_0 x} - 1} = \int_{x_c}^{\infty} dx (x_c^{\gamma-1}/x^{\gamma}) \ . \qquad (24)$$

Equation (24) gives the exponent $\gamma = 1 - 1/\ln(p_0 x_c) = 1^+$ in the limit $x_c \to 0$.

It might be noted that $P(x)$ as given by (23) is, formally, a Bose-Einstein-type occupation number (in two dimensions) for an exponential, Boltzmann-type, distribution $p(x)$. The self-replication equation (21), which describes a geometric series, has also a formal resemblance to Dyson's equation in the theory of interacting many-body ensembles. Distributions $P(x)$ as given by Euler's transform (22) can be considered for a general form of generating distributions $p(x)$, which amounts to including only the self-replication process for the accompanying events produced by $p(x) = p_0 r(x)$. For this general case, the series expansion $p(x) = p_0 - p_1 x...$ can be considered in the neighbourhood of $x = 0$, leading to Omori's law $P(x) = p_0 x_0/x$ for $x \ll x_0 = p_0/p_1$. Euler's transform (22) provides a general representation $P(x) = p_0/h(x)$ for such singular distributions, where $h(0) = 0$ and $h(\infty) \to \infty$ (such that, preferrably, $P(x)$ is integrable at infinite). It implies $p(x) = p_0(1 - h) \simeq p_0/(1 + h)$ for $x \to 0$. Such a representation may be regarded as a generalized Omori-type distribution. Equation (23) gives, actually, such a generalized Omori's law. For $h(x) \sim x^{\gamma}$, $\gamma > 0$, power-law distributions $P(x) \sim 1/x^{\gamma}$ are obtained (an upper-bound cutoff D is necessary for $0 < \gamma \leq 1$, as well as a lower-bound cutoff x_c for $1 \leq \gamma$).

References

Bak, P., Christensen, K., Danon, L. and Scanlon, T., 2002 , Unified scaling law for earthquakes, *Phys. Rev. Lett.* **88** 178501(-1-4).

Bath, M., 1965, Lateral inhomogeneities of the upper mantle, *Tectonophysics*, **2** 483-514.

Bowman, D. D., Ouillon, G., Sammis, C. G., Sornette, A. and Sornette, D., 1998, An observational test of the critical earthquake concept,*J. Geophys. Res.* **103** 24359-24372.

Buffe, C. G. and Varnes, D. J., 1993, Predictive modeling of the seismic cycle of the Greater San Francisco Bay region, *J. Geophys. Res.* **98** 9871-9888.

Bullen, K. E., 1963, *An Introduction to the Theory of Seismology*, Cambridge University Press, London.

Console, R., Lombardi, A. M., Murru, M. and Rhoades, D., 2003, Bath's law and the self-similarity of earthquakes, *J. Geophys. Res.* **108** 2128.

Gutenberg, B. and Richter, C. F., 1944, Frequency of earthquakes in California, *Bull. Seismol. Soc. Amer.* **34** 185.

Gutenberg, B. and Richter, C. F., 1954, *Seismicity of the Earth and Associated Phenomena*, Princeton.

Helmstetter, A. and Sornette, D., 2002, Diffusion of epicentres of earthquake aftershocks, Omori's law, and generalized continuos-time random walk models, *Phys. Rev.* **E66** 061104(-1-24).

Helmstetter, A., Hergarten, S. and Sornette, D., 2004, Properties of foreshocks and aftershocks of the non-conservative self-organized critical Olami-Feder-Christensen model, *Phys. Rev.* **E70** 046120(-1-13).

Kagan, Y. Y., 1999, Universality of the seismic moment-frequency relation, *Pure and Appl. Geophys.* **155** 537-573.

Kanamori, H. and Anderson, D. L., 1975, Theoretical basis of some empirical relations in seismology, *Bull. Seismol. Soc. Amer.* **65** 1073.

Omori, F., 1894, *J. Coll. Sci. Imper. Univ. Tokyo* **7** 111.

Papazachos, B. C., 1975, Foreshocks and Earthquake prediction, *Tectonophysics* **28** 213-226.

Rabinovitch, A., Frid, V. and Bahat, D., 2001, Gutenberg-Richter-type relation for laboratory fracture-induced electromagnetic radiation, *Phys. Rev.* **E65** 011401(-1-4).

Romanian Earthquake Catalogue, 2005 (updated), Institute for Earth's Physics, Magurele-Bucharest, Romania, http:www.infp.ro/eqsinfo.php/ (continuously updated Catalogue first published by Oncescu, M.C., Rizescu, V.I. and Popa, M., 1999, in *Vrancea Earthquakes and Tectonics, Hazard and Mitigation*, Wenzel, F. and Lungu, D. (eds)& Novak, O. (co-ed), pp.43-47, Kluwer, Dordrecht, Netherlands.

Sornette, D., 1998, Discrete-scale invariance and complex dimensions, *Phys. Reps.* **297** 239-270.

Sornette, D., 2003, Critical market crashes, *Phys. Reps.* **378** 1-98.

Sornette, D., Vanneste, C. and Knopoff, L., 1992, Statistical model of earthquake foreshocks, *Phys. Rev.* **A45** 8351-8357.

Sotolongo-Costa, O. and Posada, A., 2004, Fragment-asperity interaction model for earthquakes, *Phys. Rev. Lett.* **92** 048501(-1-4).

Utsu, T., 1961, *Geophys. Mag.* **30** 521-605.

Vere-Jones, D., 1969, A note on the statistical interpretation of Bath's law, *Bull. Seismol. Soc. Amer.* **59** 1535.