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Scaling and universal power laws

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Abstract

Universal power laws are derived directly from their scaling properties. Though the results are generally valid, we make the discussion herein in connection with the pair distribution of the nearest-neighbouring seismic events in temporal series.

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Recently, a great deal of attention is being given[1]-[6] to the pair distribution $D(\tau)$ of nearest-neighbouring seisms separated by time τ. In general, the probability density of N serial events denoted by i, occurring at time t_i, can be written as

$$\rho(t) = \frac{1}{N}\sum_i \delta(t_i - t) \ , \tag{1}$$

and, similarly, the pair distribution of nearest-neighbours separated by time τ is given by

$$D(\tau) = \frac{1}{N}\sum_i \delta(t_{i+1} - t_i - \tau) \ . \tag{2}$$

A seismic event occurring at time t_i has a magnitude M_i, and much attention has been given to the effect of the lower-bound cutoff magnitude M_c upon the form of such distributions. In particular, it was recently claimed[2, 3] that the pair distribution $D(\tau)$ can be represented as

$$D(\tau) = Rf(R\tau) \ , \tag{3}$$

where $R = t_0^{-1}e^{-\beta M_c}$ is the rate of seisms with magnitude greater than M_c, and f is a universal function, in the sense that it does not depend on M_c, at least. The parameters t_0^{-1} (which may be viewed as the seismicity rate) and β in the cutoff parameter R given above are well documented from various statistical analysis of earthquakes,[7]-[9] their particular values being, however, immaterial for the present discussion.

It is easy to see that the pair distribution given by (3) satisfies the scaling relationship

$$\widetilde{D}(\tau) = e^{-\beta M_c'}f(e^{-\beta M_c'}\tau) = pD(p\tau) \ , \tag{4}$$

where $\widetilde{D}(\tau)$ is the transform of $D(\tau)$ under the change of the cutoff, as given by parameter $p = e^{-\beta(M_c'-M_c)}$. Renormalization group arguments have been put forward recently[1] in order to determine the function $D(\tau)$. We show herein that it can be derived directly from the scaling equation (4), whose elementary solutions for its Laplace transform are universal power laws.

Indeed, $\widetilde{D}(\tau)$ may actually be viewed as a function $\widetilde{D}(p,\tau)$ of both p and τ, and equation (4) can also be written as

$$p^{-1}\widetilde{D}(p,\tau/p) = D(\tau) \ . \tag{5}$$

It is more convenient to write the above equation by its Laplace transform as

$$\widetilde{D}(p,ps) = D(s) \ . \tag{6}$$

According to equation (6) the derivative of $\widetilde{D}$ with respect to p must vanish. Making use of the new variable $u = ps$ we obtain

$$p\partial\widetilde{D}(p,u)/\partial p + u\partial\widetilde{D}(p,u)/\partial u = 0 \ . \tag{7}$$

By separation of variables we get the elementary solution

$$\widetilde{D}(p,u) = p^{\beta}u^{-\beta} \ , \tag{8}$$

or, by (6),

$$D(s) = s^{-\beta} \ , \tag{9}$$

where β is a real exponent. This is a universal power law. It corresponds to a power-law function $D(\tau) \sim 1/\tau^{1-\beta}$ for $\beta > 0$.

In general, the solution of equation (7) is a superposition of elementary solutions of type $s^{-\beta}$. Such a particular superposition for integer $\beta < 0$ (which is related to the hypergeometric function) may give $D(s) = (1+s)^{-\alpha}$, which corresponds to $D(\tau) \sim (1/\tau^{1-\alpha})e^{-\tau}$. The normalizing constant for this probability distribution is $1/\Gamma(\alpha)$, where Γ is Euler's gamma function. This probability distribution corresponds to $D(\tau) = RC\cdot[1/(R\tau)^{1-\alpha}]e^{-R\tau/B}$ discussed in Refs.1-3, where $B = 1.58$, $C = 1/2$ and $\alpha = 0.66$, such that $CB^{\alpha}\Gamma(\alpha) = 1$. The exponential form for large values of τ is in fact expected from the uncorrelated seismicity,[10] while the power law for small values of τ indicates a clustering process.[11]

Apart from the scaling law (3),[4] another main issue in this connection[1]-[4] resides in the particular values assumed by the exponent α and constant B in the universal function $f(\tau) = (1/\tau^{1-\alpha})e^{-\tau/B}$. It was shown[12] that seismic energy E can be accumulated in a seismic focus over time t, such that $t \sim E^r$, where r is a parameter related to the geometry of the focal zone and the focal mechanism. For a uniform mechanism of accumulating energy in a localized, point-like seismic focus the parameter r acquires the value $r = 1/3$. This parameter contributes to the exponent $\beta = br$ in the magnitude distribution of the earthquakes, for instance in the excedence rate $N_{ex}/T = t_0^{-1}e^{-\beta M}$ of earthquakes with magnitude greater than M (recurrence law), where t_0^{-1} is the seismicity rate over period T and $b = 3.5$ is the Gutenberg-Richter coefficient in the energy-magnitude relationship. For $M = M_c$ this excedence rate is the cutoff parameter R. If energy E is released in time τ, then its rate is given by $E \sim 1/\tau$ for short times τ, according to Omori's law,[13]-[15] so that the above relationship implies $t \sim E^r \sim 1/\tau^r$. On the other hand, it is natural to assume that the pair distribtion is proportional to time t, so we are led to conjecture the form

$$D(\tau) \sim \frac{1}{\tau^r} \tag{10}$$

for the pair distribution for small values of τ, which implies the exponent parameter $\alpha = 1 - r$ in the universal function $f(\tau)$. For $r = 1/3$ ($\beta = 1.17$) we get $\alpha = 2/3$ in agreement with the value indicated in Refs. 1-3.

The excedence rate given above is well documented for a variety of regions, time intervals and magnitudes. For instance, a worldwide analysis for $5.8 < M < 7.3$ indicates $r = 0.39$ ($\beta = 1.38$;

and $-\ln t_0 = 12.65$ for t_0 measured in years).[16] Data for Southern California[5] seem to indicate $r = 0.66$ ($\beta = 2.3$; and $-\ln t_0 = 17.25$ for t_0 in years). Similarly, a recent analysis for Vrancea earthquakes[17] indicates $r = 0.54$ ($\beta = 1.89$; and $-\ln t_0 = 9.68$ for t_0 in years). This rather limited variability in the exponents of the pair distribution for small values of τ seems to be supported by data.[2]-[4] In particular, the analysis of the pair distribution for 1999 earthquakes with magnitude $M > 3$ recorded in Vrancea between 1974 and 2004 indicates an exponent $\alpha = 0.75$, corresponding to $r = 0.25$, in fair agreement with the exponent $r = 0.33$[1]-[3]. The exponent r in (10) exhibits a tendency toward higher values in the limit $R\tau \to 0$, where Omori's law with exponent unity is more effective. For instance, Vrancea data indicate $r = 0.57$ in fair agreement with the value of this exponent obtained from fitting the recurrence law ($r = 0.54$).[17]

The origin of the parameter B in the universal function $f(\tau)$ seems to reside in the fact that for large values of τ (where this parameter is effective), *i.e.* for small values of t (and E), the actual relatiosnhip between t and E is $1 + t/t_0 = (1 + E/E_0)^r$, which gives a recurrence law $N_{ex}/T = t_0^{-1}(1 + e^{bM})^{-r}$ (where E_0 is the threshold energy in the Gutenberg-Richter relationship).[12] This modified recurrence law implies an apparent decrease in the cutoff parameter R from $R = t_0^{-1}e^{-\beta M_c}$ to $R = t_0^{-1}(1 + e^{bM})^{-r}$ (where $\beta = br$), which amounts, for $M = 0$, for instance, to a correction factor ~ 1.26. It is this parameter B which may account for such correction factors. For instance, the data analysis for Vrancea indicates $B = 1.17$ ($C = 0.71$), which indeed compares well with $2^r = 1.19$ corresponding to $r = 0.25$. In general, it is worth noting that exponent $1 - \alpha = r$ in the class of functions $D(\tau) = RC \cdot [1/(R\tau)^{1-\alpha}]e^{-R\tau/B}$ is a fitting parameter.

In conclusion, the scaling equation (3) indicates the general form of the universal function of the pair distribution for the nearest-neighbouring earthquakes, which consists of a certain superposition of elementary power laws, while the parameters in this universal function, namely the exponent parameter α and parameter B, are related to particular seismicities, exhibiting a relatively limited variability, such that it may be said that the pair distribution is rather represented by a quasi-universal function.

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