Journal of Theoretical Physics

Founded and Edited by M. Apostol **113** (2005)

ISSN 1453-4428

"Rule 30" and its equation

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The cellular automata reproduce a great deal of complex forms and processes, following "simple rules".[1] A typical instance is provided by "Rule 30", shown in Fig. 1. The striking phenomenon exhibited in Fig. 1 is represented by the two distinct patterns in regions I and, respectively, II, separated by a sharp boundary. In region I the picture shows striations, more or less regular and possibly random, while in region II the picture exhibits a random distribution of black and white cells. Although "Rule 30" contains a left-right assymetry, such a separating boundary is still not easy to be expected.

Let a, b, c be three successive cells at time $t_i = i$, corresponding to coordinates $x_{j-1} = j-1$, $x_j = j$ and, respectively, $x_{j+1} = j+1$. Coordinate x is measured along the horizontal axis, and time t goes downward along the vertical axis. Labels i and j assume integer values. Variables a, b, c take each two values, 1 (black) and 0 (white). "Rule 30" is defined by $a(1-b)(1-c)$, $(1-a)bc$, $(1-a)b(1-c)$, $(1-a)(1-b)c \rightarrow 1$ for b at time $t_{i+1} = i+1$, and zero otherwise. The four products above are distinct, so their sum $a+b+c-2ab-2ac-bc+2abc$ gives b for t_{i+1} and $x_j = j$. We introduce the function $u(i,j) = b$ (for $t_i = i$ and $x_j = j$), so "Rule 30" can be written as

$$\begin{aligned} u(i+1,j) = u(i,j-1) + u(i,j) + u(i,j+1) - 2u(i,j-1)u(i,j) - \\ -2u(i,j-1)u(i,j+1) - u(i,j)u(i,j+1) + 2u(i,j-1)u(i,j)u(i,j+1) \ . \end{aligned} \quad (1)$$

Solving this equation with the initial condition $u(0,0) = 1$ and $u(0, j \neq 0) = 0$ leads to the picture shown in Fig. 1.

Let us assume u be a continuous and differentiable function with values in the range 0 to 1, and write down the continuum version of equation (1):

$$\begin{aligned} \partial u/\partial t + \tfrac{1}{2}\partial^2 u/\partial t^2 + ... = 2u - 5u^2 + 2u^3 + u\partial u/\partial x + \\ +2(1-u)(\partial u/\partial x)^2 + (1 - 7u/2 + 2u^2)\partial^2 u/\partial x^2 + ... \end{aligned} \quad (2)$$

First, it is worth noting that (2) is not a "simple" non-linear equation, albeit "Rule 30" is a "simple" rule. Second, we usually assume that function u in such an equation is a reasonably smooth function, so we may cut off the higher-order derivatives, or non-linearities. For instance, equation (2) can be reduced by such a procedure to

$$\partial u/\partial t = 2u - 5u^2 + 2u^3 + u\partial u/\partial x \ , \quad (3)$$

which, nevertheless, is still sufficiently intractable. It is worth noting that by the smoothness assumption we lose a lot of details, as, for instance, the regularities in region I in Fig.1. In particular,

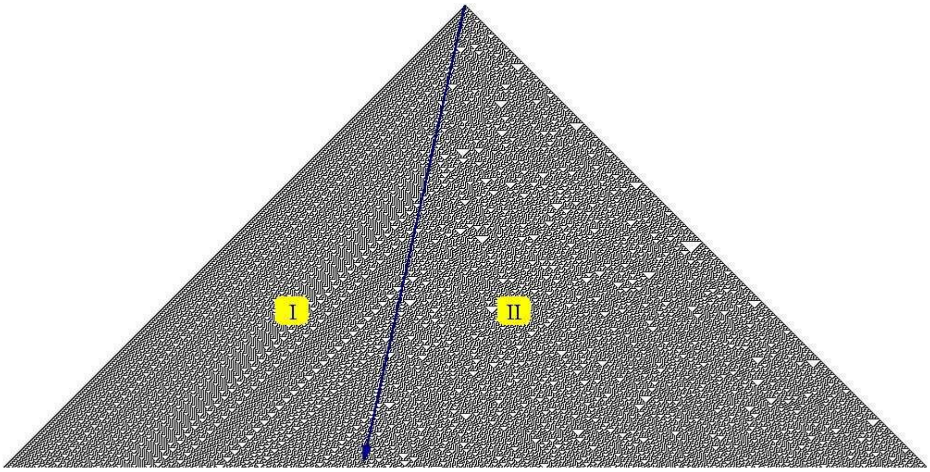

Figure 1: "Rule 30"

if we assume that u depends on $x+t$, as suggested by region I, equation (3) has a meaningless solution with respect to the original problem. The most drastic smoothness assumption is to drop out the derivatives in (3), leading to

$$2u - 5u^2 + 2u^3 = 0 \ , \tag{4}$$

whose solution $u = 1/2$ may be taken as representing a reasonable approximation to our problem.

There is an unusual feature in equation (2), that of assuming that derivatives are high with respect to other contributions, over small regions. Such a circumstance may lead to

$$\partial u/\partial t = u\partial u/\partial x \tag{5}$$

as an approximation to (2), whose solution is

$$u = -x/t \ . \tag{6}$$

The continuity requires $u = -x/t = 1/2$, which may correspond to the sharp boundary in Fig. 1 between regions I and II.

References

[1] S. Wolfram, *A New Kind of Science*, Wolfram Media, Inc. (2002)