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Kepler's problem

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Abstract

A method is described for treating small-eccentricity orbits in Kepler's problem. The method can be generalized to bound states in other central-field potentials.

Kepler's problem is the description of the motion of a particle of mass m in the gravitational potential $-\alpha/r$, where $\alpha > 0$. Like any other central potential, the gravitational potential conserves the angular momentum $\mathbf{L}$, so the motion is confined to a plane, and

$$L = mr^2\dot{\varphi} \ , \tag{1}$$

where φ is the angular coordinate. Equation (1) shows that the motion sweeps equal areas in equal times (Kepler's second law).

The energy of the motion reads

$$E = m\dot{r}^2/2 + mr^2\dot{\varphi}^2/2 - \alpha/r = m\dot{r}^2/2 + L^2/2mr^2 - \alpha/r \ , \tag{2}$$

as if the particle moves in an effective potential

$$U = L^2/2mr^2 - \alpha/r \ , \tag{3}$$

exhibiting the centrifugal $1/r^2$- energy. The closed orbits proceeds between $r_1 = a(1-e)$ and $r_2 = a(1+e)$, where $a = \alpha/2\left|E\right|$ and

$$e = \sqrt{1 - 2L^2\left|E\right|/m\alpha^2} \tag{4}$$

is the eccentricity, for negative energies above $E_{min} = -m\alpha^2/2L^2$.

The effective potential (3) reaches its minimum value E_{min} for

$$r_0 = L^2/m\alpha \ , \tag{5}$$

where the eccentricity vanishes and the orbit is circular with radius r_0. By (4), the energy can also be represented as

$$\left|E\right| = \frac{\alpha}{2r_0}(1-e^2) \ , \tag{6}$$

or $r_0 = a(1-e^2)$.

The expansion of the effective potential U given by (3) around its minimum value gives

$$U = -\frac{\alpha}{2r_0} + \frac{\alpha}{2r_0^3}(r-r_0)^2 - \frac{\alpha}{r_0^4}(r-r_0)^3 + ..., \tag{7}$$

i.e. a small-oscillations expansion valid for $|r - r_0| \ll r_0$.

It is convenient to write $r - r_0 = Au$, where u is dimensionless, and cast the energy given by (2), (3) and (7) into the form

$$E = -\alpha/2r_0 + mA^2\dot{u}^2/2 + (\alpha A^2/2r_0^3)u^2 - (\alpha A^3/r_0^4)u^3 + ..., \tag{8}$$

or

$$E = -\alpha/2r_0 + mA^2[\dot{u}^2/2 + \omega^2u^2/2 - (A/r_0)\omega^2u^3 + ...], \tag{9}$$

where $\omega^2 = \alpha/mr_0^3$ and $A/r_0 = \varepsilon$ can be viewed as a small perturbation parameter. Equation (9) can also be written as

$$e^2 = \frac{2\varepsilon^2}{\omega^2}(\dot{u}^2/2 + \omega^2u^2/2 - \varepsilon\omega^2u^3 + ...) \ , \tag{10}$$

which tells that the eccentricity e is related to the perturbation parameter ε. Equation (9) leads to the motion of an anharmonic oscillator

$$\ddot{u} + \omega^2u - 3\varepsilon\omega^2u^2 + ... = 0 \ . \tag{11}$$

Within the harmonic approximation the solution of equation (11) can be represented as $u^{(0)} = -\cos\omega t$, and

$$r^{(0)} = r_0 - A\cos\omega t \ . \tag{12}$$

The amplitude A can be derived from energy $E = -\alpha/2r_0 + mA^2\omega^2/2$ given by (9) or, equivalently, from equation (10). It leads to

$$\varepsilon = A/r_0 = e \ll 1 \ , \tag{13}$$

i.e. the eccentricity e of the orbit is the ratio ε of the amplitude A of the harmonic oscillation to the original orbit radius r_0. The small-oscillations treatment is valid for small eccentricities.

Therefore, the solution of the motion given by (12) can be written as

$$r^{(0)} = r_0(1 - e\cos\omega t) \ , \tag{14}$$

and, by (1),[1]

$$\varphi = \omega t + 2e\sin\omega t \ . \tag{15}$$

It describes a circular motion, shifted by r_0e. Indeed, $x = r_0e + r^{(0)}\cos\varphi$ and $y = r^{(0)}\sin\varphi$, such that $x^2 + y^2 = r_0^2$ within the harmonic approximation. In addition, $\omega^2 = \alpha/mr_0^3$ shows that the square of the motion period is proportional to the third power of the linear size of the orbit (Kepler's third law).[2] By (15), $\omega t = \varphi - 2e\sin\varphi$.

The first-order cubic correction to equation (11) leads to

$$u = u^{(0)} + \varepsilon u^{(1)} = -\cos\omega t - \varepsilon\cos\omega t + \frac{\varepsilon}{2}(3 - \cos 2\omega t) \ , \tag{16}$$

and equation (10) gives $\varepsilon = e(1 - e)$. The corresponding radius reads

$$r = r_0[1 - e\cos\omega t + \frac{e^2}{2}(3 - \cos 2\omega t)] \ , \tag{17}$$

[1] Noteworthy, $L = \omega I$, where $I = mr_0^2$ is the moment of inertia.
[2] J. Kepler, *Harmonices Mundi*, Linz (1619)

which, by (1), leads to

$$\varphi = \omega t + 2e\sin\omega t - \frac{e^2}{2}(3\omega t - \frac{5}{2}\sin 2\omega t)] \ . \tag{18}$$

Equation (18) can easily be inverted to give

$$\omega t = \varphi - 2e\sin\varphi + \frac{3e^2}{2}(\varphi + \frac{1}{2}\sin 2\varphi) \ , \tag{19}$$

which transforms (17) into

$$r = r_0(1 - e\cos\varphi + e^2\cos^2\varphi + ...) \ . \tag{20}$$

Within this approximation, equation (20) describes an ellipse,

$$r/r_0 = 1 - e\cos\varphi + e^2\cos^2\varphi + ... = 1/(1 + e\cos\varphi) \ , \tag{21}$$

with the semi-major axis $a = r_0/(1-e^2) = r_0(1+e^2+...)$, the semi-minor axis $b = r_0/(1-e^2)^{1/2} = r_0(1 + e^2/2 + ...)$ and the origin displaced by $ae = r_0e + ...$ in the focus ae (Kepler's first law).[3]

According to equation (19) the period T of the motion is given by

$$\omega T = 2\pi(1 + 3e^2/2) \ , \tag{22}$$

which shows that the frequency ω is shifted to $\Omega = \omega(1 - 3e^2/2) = (\alpha/ma^3)^{1/2}$. The frequency shift $\Delta\omega/\omega = -3e^2/2$ ensures the cancellation of the resonant contributions to the second-order cubic correction and first-order quartic correction to the anharmonic motion.

Let $v(r)$ be an attractive central-field potential, such that the radial motion proceeds between $r_1$and r_2 given by[4]

$$L^2/2mr_{1,2}^2 + v(r_{1,2}) = E < 0 \ . \tag{23}$$

The effective potential $U(r) = L^2/2mr^2 + v(r)$ has a minimum value $-u_0 = r_0v_1(1/2 + v_0/r_0v_1) < 0$ for r_0 given by $L^2 = mr_0^3v_1$, where v_0, v_1, v_2, ... denote the potential and, respectively, its derivatives at r_0. making use of $r - r_0 = Au$ and $A/r_0 = \varepsilon$ the energy E can be written as

$$E = -u_0 + mA^2[\dot{u}^2/2 + \omega^2u^2/2 - \varepsilon\beta\omega^2u^3 + \varepsilon^2\gamma\omega^2u^4...] \ , \tag{24}$$

where $m\omega^2 = 3v_1/r_0 + v_2$, $\beta = (2v_1 - r_0^2v_3/6)/(3v_1 + r_0v_2)$ and $\gamma = (5v_1/2 + r_0^3v_4/24)/(3v_1 + r_0v_2)$. Making use of the eccentricity e defined by $e^2 = \delta(1 - |E|/u_0)$, where $\delta = -(v_1 + 2v_0/r_0)/(3v_1 + r_0v_2)$, equation (24) can be rewritten as

$$e^2 = \frac{2\varepsilon^2}{\omega^2}(\dot{u}^2/2 + \omega^2u^2/2 - \varepsilon\beta\omega^2u^3 + \varepsilon^2\gamma\omega^2u^4 + ...) \ . \tag{25}$$

The equation of motion given by (24) reads

$$\ddot{u} + \omega^2u - 3\varepsilon\beta\omega^2u^2 + 4\varepsilon^2\gamma\omega^2u^3... = 0 \ , \tag{26}$$

and the solution is given by

$$r = r_0[1 - e\cos\omega t + \frac{\beta e^2}{2}(3 - \cos 2\omega t)] \ , \tag{27}$$

[3]Indeed, from (21), $\cos\varphi = x/(r_0 - ex)$ and $\sin\varphi = y/(r_0 - ex)$, hence the ellipse equation.

[4]In order to avoid the fall on the centre the potential $v(r)$ must be less singular at the origin than $-L^2/2mr^2$.

to the first-order of the cubic anharmonicity, where $e = \varepsilon(1 + \beta\varepsilon)$. Similarly, the angular variable is given by

$$\varphi = \sqrt{v_1/(3v_1 + r_0 v_2)}\{\omega t + 2e \sin\omega t - \frac{e^2}{2}[3(2\beta - 1)\omega t - \frac{2\beta + 3}{2}\sin 2\omega t]\} \ . \tag{28}$$

One can see that, in general, the trajectory of the motion is not closed, except for

$$\sqrt{v_1/(3v_1 + r_0 v_2)} = p/q \tag{29}$$

where p/q is a simple fraction. The gravitational potential $v(r) = -\alpha/r$ gives $p/q = 1$, while the spatial-oscillator potential $v(r) = const + \alpha r^2$ gives $p/q = 1/2$ ($\beta = 1/2$, $\gamma = 5/8$).

Denoting $1/\nu = \sqrt{v_1/(3v_1 + r_0 v_2)}$ and introducing the new phase $\chi = \nu\varphi$ equation (28) can be rewritten as

$$\chi = \omega t + 2e \sin\omega t - \frac{e^2}{2}[3(2\beta - 1)\omega t - \frac{2\beta + 3}{2}\sin 2\omega t] \ , \tag{30}$$

and it can easily be inverted to give

$$\omega t = \chi - 2e \sin\chi + \frac{e^2}{2}[3(2\beta - 1)\chi - \frac{2\beta - 5}{2}\sin 2\chi] \ . \tag{31}$$

Making use of (31) the equation of the trajectory (27) becomes

$$r = r_0'[1 - e\cos\chi + (2 - \beta)e^2\cos^2\chi] \ , \tag{32}$$

where $r_0' = r_0[1 - 2(1 - \beta)e^2]$. For the gravitational potential $\beta = 1$ and equation (21) is recovered from (32), while for the spatial oscillator $\beta = 1/2$, $\chi = 2\varphi$ and (32) becomes

$$r = r_0'[1 - e\cos 2\varphi + \frac{3e^2}{2}\cos^2 2\varphi] \ . \tag{33}$$

Since (33) is equivalent to $r^2 = r_0'^2/(1 + 2e\cos 2\varphi)$, it is easy to see that the trajectory described by (33) is an ellipse centered at the origin. One can see from (30) that the spatial oscillator does not shift the frequency, but reduces it to $\omega/2$.

Higher-order contributions of the anharmonicities may lead, in general, to a shift in frequency, in order to avoid, at each step of the perturbation calculations, the resonant terms.[5] Equation (27) for the radius gets thereby a shifted frequency ω', and equation (1) for the phase motion reads now $\dot\varphi = (L/m\omega r^2)(\omega/\omega')\omega'$, which changes, in general, the prefactor $1/\nu$ in equation (28).[6] This is valid as long as the calculations are confined to finite orders of perturbation series, as for small oscillations and eccentricities, for instance. In the limit of the series summation the orbits are closed only for two power-law potentials: the gravitational potential $-\alpha/r$ and the spatial-oscillator potential $const + \alpha r^2$. Indeed, this can be seen easily on the equation of motion

[5] Such terms are also called "secular terms", and the shift in frequency is also known as the Poincare-Lindstedt expansion (H. Poincare, *Les Methodes Nouvelles de la Mecanique Celeste*, Gauthier-Villars, Paris (1892); A. Lindstedt, *Uber die Integration einer fur die Storungstheorie wichtigen Differentialgleichung*, Astron. Nach. **103** 211 (1882)).

[6] Equation (29) is the first term of the series expansion of the well-known closure condition

$$\Delta\varphi/2\pi = (1/\pi)\int_{r_1}^{r_2} dr \cdot (L/r^2)/\sqrt{3m(E - v(r)) - L^2/r^2} = p/q \ .$$

for the trajectory $r(\varphi)$, as given by (1) and (2), whose integration requires a quadratic form of the integrand, the only one able to lead to circular functions.[7] In general, the trajectories are closed provided the potentials are such as to cancel recurssively the frequency shifts in the formal perturbation series. However, for sufficiently large p and q, and a large number of cycles, the orbits are practically closed for any potential.

[7]Making use of the substitution $r = 1/u$ the equation for the trajectory $u(\varphi)$ reads $u'' + u = -(m/L^2)\partial v/\partial u$, whose solution is given by circular functions only for the gravitational potential $v \sim u$ and the spatial oscillator potential $v \sim 1/u^2$. This observation is called sometime "Bertrand's theorem" (J. Bertrand, *Mecanique Analytique*, Comptes Rendus, Acad. Sci. **77** 849 (1873)).