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A three-body problem: the Sun, the Earth and the Moon

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Abstract

The motion of two bodies moving around a third one kept at rest, all interacting by gravitational forces, is solved from a practical standpoint, by assuming that the two bodies stay close to each other, but far away from the third.

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the positions of two bodies of mass m_1 (Earth, $m_1 \simeq 6\times 10^{24}Kg$) and, respectively, m_2 (Moon, $m_2 \simeq 7\times 10^{22}Kg$), subjected to gravitational potentials $-Gm_0m_1/r_1$, $-Gm_0m_2/r_2$ and interacting through $-Gm_1m_2/\left|\mathbf{r}_1-\mathbf{r}_2\right|$, where $G\simeq 6.7\times 10^{-11}m^3/Kg\cdot s^2$ is the gravitational constant. The body of mass m_0 (Sun, $m_0\simeq 2\times 10^{30}Kg$) is at rest. The energy is given by

$$E = m_1\dot{\mathbf{r}}_1^2/2 + m_2\dot{\mathbf{r}}_2^2/2 - Gm_0m_1/r_1 - Gm_0m_2/r_2 - Gm_1m_2/\left|\mathbf{r}_1-\mathbf{r}_2\right| \ \ , \tag{1}$$

and the angular momentum reads

$$\mathbf{L}_{tot} = m_1\mathbf{r}_1\times\dot{\mathbf{r}}_1 + m_2\mathbf{r}_2\times\dot{\mathbf{r}}_2 \ . \tag{2}$$

It is easy to see that $\mathbf{L}_{tot}$ is conserved. Making use of the center-of-mass coordinate $\mathbf{R}=m_1\mathbf{r}_1/M+m_2\mathbf{r}_2/M$, where $M=m_1+m_2$, and the relative coordinate $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$, the angular momentum becomes

$$\mathbf{L}_{tot} = M\mathbf{R}\times\dot{\mathbf{R}} + m\mathbf{r}\times\dot{\mathbf{r}} \ \ , \tag{3}$$

where $m=m_1m_2/M$ is the relative mass. Similarly, the energy can be written as

$$E = M\dot{\mathbf{R}}^2/2 + m\dot{\mathbf{r}}^2/2 - Gm_0m_1/\left|\mathbf{R}-m_2\mathbf{r}/M\right| - Gm_0m_2/\left|\mathbf{R}+m_1\mathbf{r}/M\right| - Gm_1m_2/r\,. \tag{4}$$

Since $r\ll R$ (Sun-Earth distance $r_1\simeq 15\times 10^7Km$, Moon-Earth distance $r\simeq 380\,000Km$) it is convenient to expand the gravitational potentials in (4) in powers of $\mathbf{rR}/R^2$. Keeping only the quadrupolar contribution the energy becomes

$$E = M\dot{\mathbf{R}}^2/2 + m\dot{\mathbf{r}}^2/2 - \alpha/R - \beta/r - \gamma[3(\mathbf{rR})^2/R^2 - r^2]/R^3 \ \ , \tag{5}$$

where $\alpha = Gm_0M$, $\beta=GmM$ and $\gamma=Gm_0m/2$, or

$$E = E_1 + E_2 + \gamma v \ \ , \tag{6}$$

where

$$E_1 = M\dot{\mathbf{R}}^2/2 - \alpha/R \ \ , \ \ E_2 = m\dot{\mathbf{r}}^2/2 - \beta/r \ \ , \tag{7}$$

and

$$v = -r^2(3\cos^2\chi - 1)/R^3 \ . \tag{8}$$

The angle χ in (8) is the angle between the two vectors $\mathbf{r}$ and $\mathbf{R}$. Since $r/R \sim 3\times 10^{-3}$ for Moon-Earth-Sun (and $(r/R)^2 \sim 10^{-5}$) the interaction v may be viewed as a small perturbation, and γ in (6) may act as a formal perturbation parameter. The solutions of the equations of motion corresponding to (6) to (8) are looked for in the generic form $u = u^{(0)} + \gamma u^{(1)} + \ldots$. It is convenient now to employ polar coordinates and rewrite (7) and (8) as

$$E_1 = M\dot{R}^2/2 + MR^2(\dot{\Theta}^2 + \dot{\Phi}^2\sin^2\Theta)/2 - \alpha/R \ , \tag{9}$$

and, similarly,

$$E_2 = m\dot{r}^2/2 + mr^2(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta)/2 - \beta/r \ , \tag{10}$$

where $\cos\chi = \sin\Theta\sin\theta\cos(\Phi-\varphi) + \cos\Theta\cos\theta$ in (8). The angular momentum of the relative motion reads

$$l_x = -mr^2(\dot{\theta}\sin\varphi + \dot{\varphi}\sin\theta\cos\theta\cos\varphi) \ , \ l_y = mr^2(\dot{\theta}\cos\varphi - \dot{\varphi}\sin\theta\cos\theta\sin\varphi) \ , \\ l_z = mr^2\dot{\varphi}\sin^2\theta \ , \tag{11}$$

or $l_r = 0$, $l_\theta = -mr^2\dot{\varphi}\sin\theta$, $l_\varphi = mr^2\dot{\theta}$. Similar expressions hold for the angular momentum $\mathbf{L}$ of the center of mass, and $\mathbf{L}_{tot} = \mathbf{L} + \mathbf{l}$.[1]

Leaving aside for the moment the interaction v, equations (3), (9) and (10) describe two independent motions, each in its own gravitational potential, *i.e.* two independent Kepler's problems. Denoting by superscript (0) their relevant coordinates, the solutions of these problems can readily be written.[1] Indeed, choosing $\Theta^{(0)} = \pi/2$, the radius $R^{(0)}$ and the phase $\Phi^{(0)}$ are given by

$$R^{(0)} = R_0[1 - e_1\cos\Omega t + \frac{e_1^2}{2}(3 - \cos 2\Omega t) + \ldots] \ , \tag{12}$$

and, respectively,

$$\Phi^{(0)} = \Omega t + 2e_1\sin\Omega t - \frac{e_1^2}{2}(3\Omega t - \frac{5}{2}\sin 2\Omega t)\ldots] \ , \tag{13}$$

or

$$R^{(0)} = R_0(1 - e_1\cos\Phi^{(0)} + e_1^2\cos^2\Phi^{(0)} + \ldots) = R_0/(1 + e_1\cos\Phi^{(0)}) \ , \tag{14}$$

where $R_0 = L^{(0)2}/M\alpha$, $\Omega^2 = \alpha/MR_0^3$ and $e_1 = (1 - 2R_0|E_1|/\alpha)^{1/2}$ is the eccentricity of the elliptical orbit.[2] For vanishing eccentricities (Earth's orbit eccentricity is $e_1 \simeq 0.017$) the solutions given above read $R^{(0)} = R_0$ and $\Phi^{(0)} = \Omega t$, *i.e.* a circular trajectory (and $E_1 \simeq -\alpha/2R_0$). A similar solution holds for the relative motion, though it must be written for a tilted reference frame. Making use of (10) and (11), the radius of the circular orbit is given by $r_0 = l^{(0)2}/m\beta$, and small oscillations around this position of equilibrium can also be considered (like in (12)), with a characteristic frequency given by $\omega^2 = \beta/mr_0^3$ ($\omega \gg \Omega$).[3] Since the eccentricity $e_2 = (1 - 2r_0|E_2|/\beta)^{1/2}$ is small (Moon's orbit eccentricity is $e_2 \simeq 0.055$), the orbit can again be approximated by a circle of radius $r^{(0)} = r_0$.

In order to preserve the generality, the unperturbed m-body orbit must be rotated both by an angle φ_0 (about the z-axis) and by an angle θ_0 (about one of the x- or y-axis). The latter gives

[1] In view of the great disparity between m_1 and m_2, the center of mass is located practically on the first body (Earth, $M \simeq m_1$), and the relative motion corresponds practically to the second body (Moon, $m \simeq m_2$).

[2] From $\Omega^2 = \alpha/MR_0^3$ one can check easily the Earth's year ~ 365 days.

[3] Or $\omega = l^{(0)2}/mr_0^2$; Moon's period ~ 27 days is checked from $\omega^2 = \beta/mr_0^3$.

the inclination of the m-orbit with respect to the plane of the M- body orbit.[4] The former (φ_0-) rotation can be accounted for by changing the initial moment of time, such that we assume that time in the m-body trajectory equations is shifted with respect to time in the trajectory equations of the M-body. The θ_0-rotation (about the x-axis) leads to the new coordinates $r' = r$, and θ', φ' given by

$$\cos\theta' = \sin\theta_0 \sin\varphi \ , \ \ \tan\varphi' = \cos\theta_0 \tan\varphi \ . \tag{15}$$

One can check easily that $(d\theta'/d\varphi)^2 + (d\varphi'/d\varphi)^2 \sin^2\theta' = 1$, which expresses the conservation of the angular momentum under this rotation. By (15), one obtains

$$\begin{aligned} \varphi' &= \varphi - \tfrac{1}{4}\theta_0^2 \sin 2\varphi + ... = \omega t + 2e_2 \sin\omega t - \tfrac{1}{4}\theta_0^2 \sin 2\omega t + ..., \\ \theta' &= \pi/2 - \theta_0 \sin\varphi + ... = \pi/2 - \theta_0 \sin\omega t + ..., \\ r' &= r = r_0(1 - e_2 \cos\omega t + ...) \ . \end{aligned} \tag{16}$$

These are the zeroth order contributions $u^{(0)}$ to the general solution $u = u^{(0)} + \gamma u^{(1)} + ...$ for the m-body motion. One can check easily that they do indeed verify the unperturbed equations of motion.

The equations of motion for the m-body, as given by equations (6) to (8) and (10), read

$$\begin{aligned} m\ddot{r} - mr(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + \beta/r^2 &= 2\gamma(r/R^3)(3\cos^2\chi - 1) \ , \\ d(mr^2\dot{\theta})/dt - mr^2\dot{\varphi}^2 \sin\theta\cos\theta &= 6\gamma(r^2/R^3)\cos\chi[\sin\Theta\cos\theta\cos(\Phi - \varphi) - \cos\Theta\sin\theta] \ , \\ d(mr^2 \sin^2\theta\dot{\varphi})/dt &= 6\gamma(r^2/R^3)\cos\chi\sin\Theta\sin\theta\sin(\Phi - \varphi) \ . \end{aligned} \tag{17}$$

Since the ratio of the interaction γv to energy α/R is of the order of 10^{-7}, one may take the zeroth order approximation given by (12)-(14) for the trajectory of the M-body motion, *i.e.* the motion of the M-body is not perturbed by interaction within this approximation. Similarly, the ratio of the interaction γv to energy β/r is of the order of 10^{-3}. Consequently, only the first-order corrections in γ are retained for the motion of the m-body, as well as the linear terms in e_2 and quadratic terms in θ_0 only in the zeroth-order motion given by (16). It follows that coordinates $R = R_0$, $\Theta = \pi/2$, $\Phi = \Omega t$ and $r = r_0$, $\theta = \pi/2$, $\varphi = \omega t$ are used in the *rhs* of (17). This way, the problem amounts to a Kepler's problem in an external potential. In addition, Ω may be droped out in comparison with ω in the *rhs* of (17), since $\Omega \ll \omega$.[5] Doing so, equations (17) become

$$\begin{aligned} m\ddot{r} - mr(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + \beta/r^2 &= \gamma(r_0/R_0^3)(1 + 3\cos 2\omega t) \ , \\ d(mr^2\dot{\theta})/dt - mr^2\dot{\varphi}^2 \sin\theta\cos\theta &= 0 \ , \\ d(mr^2 \sin^2\theta\dot{\varphi})/dt &= -3\gamma(r_0^2/R_0^3)\sin 2\omega t \ . \end{aligned} \tag{18}$$

The solutions of these equations are looked for in the form $r = r' + \gamma r_1 + ...$, $\theta = \theta' + \gamma\theta_1 + ...$ and $\varphi = \varphi' + \gamma\varphi_1 + ...$, where r', θ' and φ' are given by (16). According to the present approximation mixed terms of the form $e_{1,2}\theta_0$, $e_{1,2}\theta_0^2$ are not kept in calculations, nor higher-order terms of the forms $e_{1,2}^2$, etc (but θ_0^2-terms are retained). The presence of the constant term in the first equation (18) gives rise to secular terms, so the frequency ω must to be renormalized to ω' in (16).

[4]It corresponds to Moon's orbit inclination against the ecliptic, which is approximately $\theta_0 = 5° = \pi/36$.
[5]The ratio of these Earth-Moon frequencies is $\Omega/\omega \simeq 1/13$.

This renormalization implies a shift in frequency of the order of γ, which, as it is well known, is computed by requiring the cancellation of the secular terms.

It is easy to see that equation on the third row in (18) leads to the integral of motion

$$mr^2\dot{\varphi}\sin^2\theta = F(t) + l'_z \ , \tag{19}$$

where

$$F(t) = \gamma(3r_0^2/2\omega R_0^3)\cos 2\omega t \tag{20}$$

and

$$l'_z = mr_0^2\omega'(1-\theta_0^2/2) \tag{21}$$

is a constant of integration. It is reminiscent of the z-component of the unperturbed angular momentum $l_z^{(0)}$, renormalized by γ-interaction (through frequency ω'). Equation (19) expresses the motion of the z-component of the angular momentum in the presence of the perturbation. It leads to equation

$$2m\omega r_1 + mr_0\dot{\varphi}_1 = (3r_0/2\omega R_0^3)\cos 2\omega t \tag{22}$$

for the functions r_1 and $\dot{\varphi}_1$.

Similarly, by making use of (19), equation on the second row in (18) leads to another integral of motion

$$(mr^2\dot{\theta})^2 + \frac{l_z'^2}{\sin^2\theta} = l'^2 \ , \tag{23}$$

where

$$l' = mr_0^2\omega' \tag{24}$$

is another constant of integration (reminiscent of the unperturbed angular momentum $l^{(0)}$, renormalized by γ-interaction). Equation (23) has the same form as the one corrresponding to the unperturbed motion, so it gives no equation for r_1 and θ_1, as it can be checked easily.

Finally, by making use of the two integrals of motion given by (19) and (23), the first equation in (18) leads to

$$m\ddot{r}' - l'^2/mr'^3 + \beta/r'^2 = \gamma(r_0/R_0^3) \tag{25}$$

and

$$m\ddot{r}_1 + (3l^{(0)2}/mr_0^4)r_1 - (2\beta/r_0^3)r_1 = 6(r_0/R_0^3)\cos 2\omega t \ . \tag{26}$$

Equation (25) gives the shifted frequency

$$\omega' = \omega(1-\gamma r_0^3/2\beta R_0^3) = \omega(1-\Omega^2/4\omega^2) \tag{27}$$

and the unperturbed solution r' in (16), with eccentricity e'_2 corresponding to another constant of integration E'_2 (unperturbed energy). Equations (22) and (26) can now be easily solved. Their solutions read

$$\begin{aligned} r_1 &= -(2r_0^4/\beta R_0^3)\cos 2\omega t \ , \\ \varphi_1 &= -(5r_0^3/4\beta R_0^3)\sin 2\omega t \ . \end{aligned} \tag{28}$$

The solution of the m-body motion within this approximation is now complete. It is given by (16) and by (28), with shifted frequency ω' given by (27). Within this approximation $\theta_1 = 0$. One can check that the total energy $E_2 + \gamma v = E'_2 - \gamma(r_0^2/R_0^3)$ is constant. The motion is characterized by three basic frequencies: Ω, ω and ω', though the bare frequency ω is not observable. The calculations can be extended to higher-order terms, where combined frequencies appear, as well as additional contributions to the frequency shift. The method can also be applied to other situations of three bodies interacting through gravitational potentials, like, for instance, two bodies gravitating around a third one (Jupiter and Saturn, for instance, where a natural perturbation is just their own interaction, since their mass is much lighter than Sun's mass, and they do not get too close to each other).

It is well known that Moon's orbit exhibits four periodicities, beside $T_0 \simeq 365.26$days of the year corresponding to frequency Ω. There is, first, the sideral Moon $T_1 \simeq 27.32$days, then the anomalous Moon $T_2 \simeq 27.55$days, the nodal Moon $T_3 \simeq 27.21$days and the synodal Moon $T_4 \simeq 29.53$days.[6] Making use of the numerical data given herein ($m \simeq 7\times10^{22}$Kg, $M \simeq 6\times10^{24}$Kg, $m_0 \simeq 2\times10^{30}$Kg, $r_0 \simeq 384\,000$Km, $R \simeq 150\,000$Km) and the gravitational constant $G = 6.7 \times 10^{-11} m^3/Kg \cdot s^2$, one gets easily $T_0 \simeq 364.78$days from $\Omega^2 = \alpha/MR_0^3$, and the bare period $\bar{T} \simeq 27.28$days, corresponding to the bare frequency $\omega^2 = \beta/mr_0^3$. The sideral Moon corresponds to frequency ω' given by (27), and one can check easily that it implies a frequency shift $\delta\omega/\omega = -\Omega^2/4\omega^2 \simeq -1.4 \times 10^{-3}$. It corresponds to a difference of $\delta T \simeq 0.04$days, which gives the sideral Moon $T_1 = \bar{T} + \delta T \simeq 27.32$days. In the rotating frame of the Earth the periodicity is $\omega' - \Omega$, which corresponds to a change $\delta\omega/\omega' = -\Omega/\omega \simeq 0.08$ in frequency. It implies a change $\delta T \simeq 2.2$days, corresponding to the synodal Moon $T_4 = T_1 + \delta T \simeq 29.52$days. The nodal Moon is associated with the periodiciy of the $\tilde{z}$ coordinate in the rotating frame. It is easy to see, by using directly the transcription of the hamiltonian given by (5) in the rotating frame, that this frequency is given by $\tilde{\omega}^2 = \omega^2 + \Omega^2 = \omega'^2(1 + \Omega^2/2\omega^2) + \Omega^2$, which implies a change $\delta\omega/\omega' = 3\Omega^2/4\omega^2$. It corresponds to $\delta T \simeq -0.11$days, which gives the nodal Moon $T_3 = T_1 - \delta T \simeq 27.21$days. This correction gives also $(4\omega/3\Omega)T_0 \simeq 18$years for the slow motion of Moon's nodal plane.[7] According to (16) and (28) the angle φ reads $\varphi \simeq \omega' t - (5\Omega^2/4\omega)t + ...$ in the limit of short times, which amounts to a change $\delta\omega/\omega' = -3\Omega^2/2\omega^2$ in frequency. It leads to $\delta T \simeq 0.22$days, *i.e.* a difference twice as much as the difference between the nodal Moon and the sideral Moon, which may be associated with Moon's anomaly $T_2 = T_1 + \delta T \simeq 27.54$.

In the rotating frame the motion is described by four frequencies, within this approximation, in agreement with the empirical periodicities, and with three constants of motion (energy E_2 and angular momenta l', l'_z). In general, it seems unlikely to exist additional constants of motion, beside total energy E and angular momentum $\mathbf{L}_{tot}$, as to match the six degrees of freedom, at least with analytical functions (and series), so the three-body problem is not "integrable".[8] However, non-analytical behaviour may exist, as, for instance, an infinite phase velocity $\dot{\varphi}$ for a vanishing polar angle θ. This may imply an abrupt change in the trajectory (for instance, instead of rotating very fast around the pole, the trajectory may take suddenly a longitudinal circle). Apart from particular initial conditions, such chaotic behaviour of the three-body problem would require an external perturbation, usually time dependent, like in the "Moon's problem", where Earth's coordinates act like time-dependent external fields. Nevertheless, the motion described above, very likely, by four fundamental frequencies (as well as by the corresponding "combined" frequencies and their superior harmonics), may look already very complicated to warrant the adjective "erratic", or "chaotic", though over very small a scale of magnitude.[9]

[6] These periodicities were known from ancient times, from the Greeks or even Babylonians, with five decimals, which amounts to an accuracy of one second of time. The present approximation gives a second decimal, or the first decimal, at most.

[7] Correction $3\Omega^2/4\omega^2$ was known to Newton.

[8] This is sometime referred to as a Bruns-Poincare theorem.

[9] In particular, the corresponding orbits are not closed (or "periodic") anymore.

References

[1] M. Apostol, J. Theor. Phys. **114** 1 (2005)