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A peculiar motion in Coulomb potential and a new route of quantizing the Hydrogen atom

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

The energy in Coulomb (or gravitational) potential $-\alpha/r$ reads

$$E = m\dot{r}^2/2 + L^2/2mr^2 - \alpha/r \ , \tag{1}$$

where m is the particle mass and L is the angular momentum. Let $L = 0$ for the moment, and $E = -\alpha/r_0$. The particle will pass through the origin up to the second r_0, then will return to the former r_0, in a periodic movement. Formally, it can be viewed as oscillating between $r = 0$ and $r = r_0$, around $r_0/2$. By (1), it is easy to get

$$m\dot{r}^2/2 + \alpha(r - r_0)/rr_0 = 0 \ , \tag{2}$$

or, by $r = r_0/2 + \rho$,

$$m\dot{\rho}^2/2 + \alpha(\rho - r_0/2)/r_0(\rho + r_0/2) = 0 \ . \tag{3}$$

It is also convenient to use $\rho = r_0 u/2$, so (3) becomes

$$\dot{u}^2 + \omega^2(u - 1)/(u + 1) = 0 \ , \tag{4}$$

where

$$\omega^2 = 8\alpha/mr_0^3 \ . \tag{5}$$

It is easy to integrate equation (4). One obtains

$$2\arcsin\sqrt{(1 - u)/2} + \sqrt{1 - u^2} = \omega t \ , \tag{6}$$

and the solution must be periodically extended to any t. It describes a periodic motion with period $T = 4\pi/\omega$, as if the frequency would be $\omega/2$.

Equation (3) gives also

$$m\dot{\rho}^2/2 + m\omega^2\rho^2/2 + ... - \alpha/r_0 = 0 \tag{7}$$

by expansion in powers of ρ, which describes a linear harmonic oscillator. It follows

$$\hbar\omega(n + 1/2)/2 = \alpha/r_0 \ , \tag{8}$$

where frequency $\omega/2$ is used (as for the complete motion) and $n = 0, 1, 2, ...$ One can also write it as

$$\hbar\omega\delta n/2 = |E| \ , \tag{9}$$

where $\delta n = n = 1, 2, 3...$, and making use of $\omega = (8\left|E\right|^3/m\alpha^2)^{1/2}$ one gets

$$\left|E\right| = \frac{m\alpha^2}{2\hbar^2 n^2} \ , \tag{10}$$

i.e. the quantized energy of the Hydrogen atom. The anharmonic corrections to (7) do not contribute, as it can be seen from the variation equation (9). The corresponding approximate wavefunctions of the linear oscillator must be displaced so as to be peaked on the origin.

Similarly, for $L \neq 0$, the effective potential in (1) has a minimum value for

$$r_0 = L^2/m\alpha \ , \tag{11}$$

and energy reads

$$E = m\dot{r}^2/2 + (\alpha/2r_0^3)(r - r_0)^2 + ... - \alpha/2r_0 \ . \tag{12}$$

The frequency is given by $m\omega^2 = \alpha/r_0^3$, and

$$\hbar\omega(n + 1/2)/2 = \alpha/2r_0 + E \ . \tag{13}$$

Since $L^2/2I = L^2/2mr_0^2 = \alpha/2r_0$, it is easy to see that (13) leads to

$$\sqrt{2\hbar^2\left|E\right|^3/m\alpha^2}\delta n = \left|E\right| \ , \tag{14}$$

which is again the quantal energy (10) of the Hydrogen atom. The corresponding approximate wavefunctions of linear harmonic oscillator are now peaked on r_0.

The method can be generalized to any central-force potential $v(r)$. The minimum of the effective potential is reached for r_0 given by

$$-L^2/mr_0^3 + v_1 = 0 \tag{15}$$

where $v_1$is the first derivative of v for r_0. The energy expansion reads

$$E = m\dot{r}^2/2 + (3v_1/r_0 + v_2)(r - r_0)^2/2 + ... + L^2/2mr_0^2 + v_0 \ , \tag{16}$$

where v_0 is the potential function for r_0, v_2 is the second derivative of v for r_0, and frequency is given by

$$\omega^2 = 3v_1/mr_0 + v_2/m \ . \tag{17}$$

The quantization relation reads

$$\hbar\omega(n + 1/2)/2 = E - L^2/2mr_0^2 - v_0 \ . \tag{18}$$

Making use of (15) the energy can be related to r_0 by $\left|E\right| = L^2/2mr_0^2 = v_1 r_0/2$. It follows the quantized energy is given by

$$\hbar^2[3v_1(\left|E\right|)/mr_0(\left|E\right|) + v_2(\left|E\right|)/m]n^2/4 = \left|E\right|^2 \ , \tag{19}$$

where $n = 1, 2, 3....$