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Theory of perturbation for a three-body gravitational bound state

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Abstract

A bound state of three bodies interacting through gravitational (Newton) forces is treated by means of the perturbation theory, by making use of series expansions in powers of orbit eccentricities, inclination of the orbit plane and mutual interaction. This particular three-body gravitational bound state models the Sun-Earth-Moon problem.

Introduction. The three-body problem. Let us assume three bodies of mass m_0, m_1 and m_2 interacting through gravitational (Newton) potentials. With the body of mass m_0 at the origin of the reference frame the energy reads

$$E = m_1\dot{\mathbf{r}}_1^2/2 + m_2\dot{\mathbf{r}}_2^2/2 - Gm_0m_1/r_1 - Gm_0m_2/r_2 - Gm_1m_2/\left|\mathbf{r}_1 - \mathbf{r}_2\right| \ , \tag{1}$$

where G is the gravitational constant and $\mathbf{r}_1$, $\mathbf{r}_2$ denote the positions of the two bodies of mass m_1 and, respectively, m_2. As it is well known,[1] this problem conserves the total angular momentum $\mathbf{L}_{tot} = m_1\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \times \dot{\mathbf{r}}_2$, but, in general, has no other constants of motion. Consequently, the problem is non-integrable, in general. We assume a bound state for the motion of the three bodies, with $|\mathbf{r}_1 - \mathbf{r}_2| \ll r_1, r_2$, small orbit eccentricities, and a small mutual inclination of the orbit planes. Traditionally, the problem is treated in the rotating frame of the Earth by means of Hill equation.[2]-[6] It is shown herein, that under such particular circumstances the trajectories can be computed by series expansions in powers of these small parameters. The mutual interaction between the two bodies is sufficiently weak in this case that the motions decouple from one another, and the problem reduces to an independent Kepler's problem and another Kepler's problem in an external field. Such particular situation is relevant for the Sun-Earth-Moon motion, where Sun is assigned mass m_0, Earth is assigned mass m_1 and Moon is assigned mass m_2.

We pass to the center-of-mass frame by defining $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/M$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, where $M = m_1 + m_2$. Under this transform equation (1) becomes

$$E = M\dot{\mathbf{R}}^2/2 + m\dot{\mathbf{r}}^2/2 - Gm_0m_1/\left|\mathbf{R} - m_2\mathbf{r}/M\right| - Gm_0m_2/\left|\mathbf{R} + m_1\mathbf{r}/M\right| - Gm_1m_2/r \ , \tag{2}$$

where $m = m_1m_2/M$ is the relative mass. Since $r/R \ll 1$, the interaction term in (2) can be expanded in powers of r/R, such that (2) becomes

$$E = M\dot{\mathbf{R}}^2/2 + m\dot{\mathbf{r}}^2/2 - \alpha/R - \beta/r - \gamma[3(\mathbf{r}\mathbf{R})^2/R^2 - r^2]/R^3 + ... \ , \tag{3}$$

where $\alpha = Gm_0M$, $\beta = GmM$ and $\gamma = Gm_0m/2$. It is now easy to see that E given by (3) corresponds to two Kepler's problems inter-connected by a quadrupolar γ-interaction. Higher-order multipoles may be included in (3), though their contributions are progressively small. We limit ourselves here to the quadrupolar term only. The quadrupolar contribution to (3) is of the order of $\gamma r^2/R^3$, and its ratio to α/R is $\sim (m/M)(r/R)^2 \sim 10^{-7}$ at most.[1] It follows that the interaction term affects the motion of the M-body to a very small extent only, and, consequently, we do not take into account this effect herein. The M-body motion in the present approximation is therefore described by an independent Kepler's problem. Similarly, the ratio of the γ-interaction to β/r is $\sim 10^{-3}$. We keep in the present calculations only the contributions of the first power in γ to the relative motion.[2] Under these circumstances, the m-body motion is described by a Kepler's problem in an external field brought about by the γ-interaction in (3). Within this approximation the total angular momentum is conserved up to zeroth order terms.

Making use of spherical coordinates (R, Θ, Φ) for $\mathbf{R}$ and (r, θ, ϕ) for$\mathbf{r}$, the energy (3) can be written as

$$E = E_1 + E_2 + \gamma v \ , \tag{4}$$

where

$$\begin{aligned} E_1 &= M\dot{\mathbf{R}}^2/2 - \alpha/R = M\dot{R}^2/2 + (MR^2/2)(\dot{\Theta}^2 + \dot{\Phi}^2 \sin^2\Theta) - \alpha/R \ , \\ E_2 &= m\dot{\mathbf{r}}^2/2 - \beta/r = m\dot{r}^2/2 + (mr^2/2)(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) - \beta/r \end{aligned} \tag{5}$$

describe two independent Kepler's problems, the interaction v reads

$$v = -r^2(3\cos^2\chi - 1)/R^3 \ , \tag{6}$$

and the angle χ between the two vectors $\mathbf{R}$ and $\mathbf{r}$ is given by

$$\cos\chi = \sin\Theta \sin\theta \cos(\Phi - \varphi) + \cos\Theta \cos\theta \ . \tag{7}$$

The general procedure we follow here is to write down the equations of motion for (R, Θ, Φ) and (r, θ, ϕ) by making use of (4) to (7), and solve them by means of the perturbation theory by writing solutions as $u = u^{(0)} + \gamma u^{(1)} + ...$, where $u^{(0)}$ denotes the generic solutions of Kepler's problem. Though Kepler' s problem has solutions in closed forms, they are not convenient for integrating the u_1-equations. Consequently, we cast first the solution of Kepler's problem into a series expansion in powers of eccentricities, and also in powers of the inclination angle of the $\mathbf{r}$-motion plane with respect to the orbit plane of the $\mathbf{R}$-motion.

Kepler's problem. As it is well known,[7, 8] the energy E_2 given by (5) can be written as

$$E_2 = m\dot{r}^2/2 + U(r) \ , \tag{8}$$

where the effective potential $U(r) = l^2/2mr^2 - \beta/r$ includes the centrifugal term $l^2/2mr^2$, $l = mr^2\dot{\varphi}$ being the (conserved) angular momentum. The effective potential $U(r)$ has a minimum value $E_{min} = -m\beta^2/2l^2 = -\beta/2r_0$ reached for $r_0 = l^2/m\beta$. We expand the effective potential $U(r)$ in powers of $r - r_0 = \varepsilon r_0 u$ around this minimum, where u is a dimensionless coordinate and ε is a small parameter related to energy (and the eccentricity of the orbit). Doing so, the energy E_2 given by (8) becomes

$$E_2 = -\beta/2r_0 + m\varepsilon^2 r_0^2[\dot{u}^2/2 + \omega^2 u^2/2 - \varepsilon\omega^2 u^3 + ...] \ , \tag{9}$$

[1]We take approximately $m \simeq 7 \times 10^{22}$Kg (Moon's mass), $M \simeq 6 \times 10^{24}$Kg (Earth's mass), and $r \simeq 384\,000$Km (Earth-Moon distance), $R \simeq 150 \times 10^6$Km (Earth-Sun distance). Sun's mass is approximately $m_0 \simeq 2 \times 10^{30}$Kg.

[2]As it is well known, Moon's periodicities are known up to five decimals (accuracy of one second of time).[1] The present approximation amounts to second-decimal and the first-decimal contributions to these periodicities, at most.

where frequency ω is given by $\omega^2 = \beta/mr_0^3$. It is easy to see that equation (9) can also be written as

$$e_2^2 = \frac{2\varepsilon^2}{\omega^2}(\dot{u}^2/2 + \omega^2 u^2/2 - \varepsilon\omega^2 u^3 + ...) \ , \tag{10}$$

where $e_2 = \sqrt{1 - 2l^2 |E_2| / m\beta^2} = \sqrt{1 - 2r_0 |E_2| / \beta}$ is the eccentricity of the orbit, for any $E_{min} < E_2 < 0$. The equation of motion given by (9) corresponds to an anharmonic oscillator, and the solution can be obtained by a perturbation expansion in powers of ε, of the form $u = u^{(0)} + \varepsilon u^{(1)} + ...$. It leads to $\varepsilon = e_2(1 - e_2)$ to the second-order in eccentricity $e_2 \ll 1$, and the trajectory equation

$$r = r_0[1 - e_2 \cos\omega t + \frac{e_2^2}{2}(3 - \cos 2\omega t) + ...] \ . \tag{11}$$

Making use of $l = mr^2\dot{\varphi}$ we obtain also the time dependence

$$\varphi = \omega t + 2e_2 \sin\omega t - \frac{e_2^2}{2}(3\omega t - \frac{5}{2}\sin 2\omega t) + ...] \ . \tag{12}$$

of the angular variable φ. Eliminating time between (11) and (12) we obtain the well known parametric equation of the trajectory

$$r = r_0(1 - e_2\cos\varphi + e_2^2\cos^2\varphi + ...) = r_0/(1 + e_2\cos\varphi) \ , \tag{13}$$

which describes an ellipse of parameter r_0 and eccentricity e_2, with the semi-major axis $a = r_0/(1 - e_2^2) = r_0(1 + e_2^2 + ...)$, the semi-minor axis $b = r_0/(1 - e_2^2)^{1/2} = r_0(1 + e_2^2/2 + ...)$ and the origin displaced by $ae_2 = r_0 e_2 + ...$ in the focus ae_2.

Solutions of the type given by (11) and (12) are used herein for the unperturbed equations of motion correponding to the three-body problem given by (4).[3] For the motion of the M-body they read

$$\begin{aligned} R &= R_0[1 - e_1\cos\Omega t + \tfrac{e_1^2}{2}(3 - \cos 2\Omega t) + ...] \ , \\ \Phi &= \Omega t + 2e_1\sin\Omega t - \tfrac{e_1^2}{2}(3\Omega t - \tfrac{5}{2}\sin 2\Omega t) + ...] \ . \end{aligned} \tag{14}$$

where the fequency Ω is now given by $\Omega^2 = \alpha/MR_0^3$, the orbit parameter is given by $R_0 = L^2/m\alpha$, L is the corresponding angular momentum and e_1 is the corresponding eccentricity. We perform the present calculations up to the first order in eccentricities $e_{1,2}$.[4] As it was said above, the γ-interaction does not affect the solution given by (14) within this order of approximation. In addition, we choose the orbit plane of this motion as defined by angle $\Theta = \pi/2$.

The situation is different for the unperturbed motion of the m-body. In order to preserve the generality, the unperturbed m-body orbit must be rotated both by an angle φ_0 (about the z-axis) and by an angle θ_0 (about one of the x- or y-axis). The latter gives the inclination of the m-orbit with respect to the plane of the M- body orbit.[5] The former (φ_0-) rotation can be accounted for by changing the initial moment of time, such that we assume that time in the m-body trajectory equations given by (11) and (12) is shifted with respect to time in the trajectory equations of the M-body, as given by (14). The θ_0-rotation (about the x-axis) leads to the new coordinates $r' = r$, and θ', φ' given by

$$\cos\theta' = \sin\theta_0 \sin\varphi \ , \ \ \tan\varphi' = \cos\theta_0 \tan\varphi \ . \tag{15}$$

[3]The method of expansion in powers of eccentricities can also be used for other types of central-field potentials.

[4]Earth's orbit eccentricity is approximately $e_1 \simeq 0.017$ and Moon's orbit eccentricity is approximately $e_2 \simeq 0.055$. The orbits are displaced circles within this approximation.

[5]It corresponds to Moon's orbit inclination against the ecliptic, which is approximately $\theta_0 = 5^\circ = \pi/36$.

One can check easily that $(d\theta'/d\varphi)^2 + (d\varphi'/d\varphi)^2 \sin^2\theta' = 1$, which expresses the conservation of the angular momentum under this rotation. We limit ourselves to contributions up to θ_0^2-order in (15), such that

$$\varphi' = \varphi - \tfrac{1}{4}\theta_0^2 \sin 2\varphi + ... = \omega t + 2e_2 \sin\omega t - \tfrac{1}{4}\theta_0^2 \sin 2\omega t + ...,$$
$$\theta' = \pi/2 - \theta_0 \sin\varphi + ... = \pi/2 - \theta_0 \sin\omega t + ..., \tag{16}$$
$$r' = r = r_0(1 - e_2\cos\omega t + ...) \ .$$

These are the zeroth order contributions $u^{(0)}$ to the general solution $u = u^{(0)} + \gamma u^{(1)} + ...$ for the m-body motion. One can check easily that they do indeed verify the unperturbed equations of motion.

Equations of motion. The equations of motion for the m-body, as given by (4) to (7), read

$$m\ddot{r} - mr(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \beta/r^2 = 2\gamma(r/R^3)(3\cos^2\chi - 1) \ ,$$
$$d(mr^2\dot{\theta})/dt - mr^2\dot{\varphi}^2 \sin\theta\cos\theta = 6\gamma(r^2/R^3)\cos\chi[\sin\Theta\cos\theta\cos(\Phi - \varphi) - \cos\Theta\sin\theta] \ , \tag{17}$$
$$d(mr^2\sin^2\theta\dot{\varphi})/dt = 6\gamma(r^2/R^3)\cos\chi\sin\Theta\sin\theta\sin(\Phi - \varphi) \ .$$

Within the present approximation, we insert in the *rhs* of (17) $R = R_0$, $\Theta = \pi/2$, $\Phi = \Omega t$, as given by the zeroth order terms in (14), and $r = r_0$, $\theta = \pi/2$, $\varphi = \omega t$, according to similar terms in (16). In addition, we may drop out Ω in comparison with ω in the *rhs* of (17), since we assume that $\Omega \ll \omega$.[6] Doing so, equations (17) become

$$m\ddot{r} - mr(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \beta/r^2 = \gamma(r_0/R_0^3)(1 + 3\cos 2\omega t) \ ,$$
$$d(mr^2\dot{\theta})/dt - mr^2\dot{\varphi}^2\sin\theta\cos\theta = 0 \ , \tag{18}$$
$$d(mr^2\sin^2\theta\dot{\varphi})/dt = -3\gamma(r_0^2/R_0^3)\sin 2\omega t \ .$$

We look for solutions of these equations in the form $r = r' + \gamma r_1 + ...$, $\theta = \theta' + \gamma\theta_1 + ...$ and $\varphi = \varphi' + \gamma\varphi_1 + ...$, where r', θ' and φ' are given by (16). According to the present approximation we do not keep in calculations mixed terms of the form $e_{1,2}\theta_0$, $e_{1,2}\theta_0^2$, nor higher-order terms of the forms $e_{1,2}^2$, etc (but we keep θ_0^2-terms). In addition, the presence of the constant term in the first equation (18) gives rise to secular terms, so we renormalize the frequency $\omega \to \omega'$ in the zeroth order solutions given by (16). This renormalization implies a shift in frequency of the order of γ, which, as it is well known, is computed by requiring the cancellation of the secular terms.

It is easy to see that equation on the third row in (18) leads to the integral of motion

$$mr^2\dot{\varphi}\sin^2\theta = F(t) + l_z' \ , \tag{19}$$

where

$$F(t) = \gamma(3r_0^2/2\omega R_0^3)\cos 2\omega t \tag{20}$$

and

$$l_z' = mr_0^2\omega'(1 - \theta_0^2/2) \tag{21}$$

[6]The ratio of these Earth-Moon frequencies is $\Omega/\omega \simeq 1/13$.

is a constant of integration. It is reminiscent of the z-component of the unperturbed angular momentum, renormalized by γ-interaction (through frequency ω'). Equation (19) expresses the motion of the z-component of the angular momentum in the presence of the perturbation. It leads to equation

$$2m\omega r_1 + mr_0\dot{\varphi}_1 = (3r_0/2\omega R_0^3)\cos 2\omega t \tag{22}$$

for the functions r_1 and $\dot{\varphi}_1$.

Similarly, by making use of (19), equation on the second row in (18) leads to another integral of motion

$$(mr^2\dot{\theta})^2 + \frac{l_z'^2}{\sin^2\theta} = l'^2 \ , \tag{23}$$

where

$$l' = mr_0^2\omega' \tag{24}$$

is another constant of integration (reminiscent of the unperturbed angular momentum, renormalized by γ-interaction). Equation (23) has the same form as the one corrresponding to the unperturbed motion, so it gives no equation for r_1 and θ_1, as it can be checked easily.

Finally, by making use of the two integrals of motion given by (19) and (23), the first equation in (18) leads to

$$m\ddot{r}' - l'^2/mr'^3 + \beta/r'^2 = \gamma(r_0/R_0^3) \tag{25}$$

and

$$m\ddot{r}_1 + (3l^2/mr_0^4)r_1 - (2\beta/r_0^3)r_1 = 6(r_0/R_0^3)\cos 2\omega t \ . \tag{26}$$

Equation (25) gives the shifted frequency

$$\omega' = \omega(1 - \gamma r_0^3/2\beta R_0^3) = \omega(1 - \Omega^2/4\omega^2) \tag{27}$$

and the unperturbed solution r' in (16), with eccentricity e_2' corresponding to another constant of integration E_2' (unperturbed energy). Equations (22) and (26) can now be easily solved. Their solutions read

$$\begin{aligned} r_1 &= -(2r_0^4/\beta R_0^3)\cos 2\omega t \ , \\ \varphi_1 &= -(5r_0^3/4\beta R_0^3)\sin 2\omega t \ . \end{aligned} \tag{28}$$

The solution of the m-body motion within this approximation is now complete. It is given by (16) and by (28), with shifted frequency ω' given by (27). Within this approximation $\theta_1 = 0$. One can check that the total energy $E_2 + \gamma v = E_2' - \gamma(r_0^2/R_0^3)$ is constant. The motion is characterized by three basic frequencies: Ω, ω and ω', though the bare frequency ω is not observable. The calculations can be extended to higher-order terms, where combined frequencies may appear, as well as additional contributions to the frequency shift.

Conclusions. It is well known that Moon's orbit exhibits four basic periodicities, beside $T_0 \simeq$ 365.26days of the year corresponding to frequency Ω. There is, first, the sideral Moon $T_1 \simeq$ 27.32days, then the anomalous Moon $T_2 \simeq 27.55$days, the nodal Moon $T_3 \simeq 27.21$days and the synodal Moon $T_4 \simeq 29.53$days.[1] Making use of the numerical data given herein ($m \simeq 7\times10^{22}$Kg, $M \simeq 6\times10^{24}$Kg, $m_0 \simeq 2\times10^{30}$Kg, $r_0 \simeq 384\,000$Km, $R \simeq 150\,000$Km) and the gravitational constant $G = 6.7\times10^{-11}m^3/Kg\cdot s^2$, we get easily $T_0 \simeq 364.78$days from $\Omega^2 = \alpha/MR_0^3$, and the bare period $\bar{T} \simeq 27.28$days, corresponding to the bare frequency $\omega^2 = \beta/mr_0^3$. The sideral Moon corresponds to frequency ω' given by (27), and one can check easily that it implies a frequency shift $\delta\omega/\omega = -\Omega^2/4\omega^2 \simeq -1.4\times10^{-3}$. It corresponds to a difference of $\delta T \simeq 0.04$days, which gives the sideral Moon $T_1 = \bar{T} + \delta T \simeq 27.32$days. In the rotating frame of the Earth the periodicity is given by $\omega' - \Omega$, which corresponds to a change $\delta\omega/\omega' = -\Omega/\omega \simeq 0.08$ in frequency.[7] It implies a change $\delta T \simeq 2.2$days, corresponding to the synodal Moon $T_4 = T_1 + \delta T \simeq 29.52$days. The nodal Moon is associated with the periodiciy of the $\tilde{z}$ coordinate in the rotating frame. It is easy to see, by using directly the transcription of the hamiltonian given by (3) in the rotating frame, that this frequency is given by $\tilde{\omega}^2 = \omega^2 + \Omega^2 = \omega'^2(1 + \Omega^2/2\omega^2) + \Omega^2$, which implies a change $\delta\omega/\omega' = 3\Omega^2/4\omega^2$. It corresponds to $\delta T \simeq -0.11$days, which gives the nodal Moon $T_3 = T_1 - \delta T \simeq 27.21$days. This correction gives also $(4\omega/3\Omega)T_0 \simeq 18$years for the slow motion of Moon's nodal plane. According to (16) and (28) the angle φ reads $\varphi \simeq \omega' t - (5\Omega^2/4\omega)t + ...$ in the limit of short times, which amounts to a change $\delta\omega/\omega' = -3\Omega^2/2\omega^2$. It leads to $\delta T \simeq 0.22$days, *i.e.* a difference twice as much as the difference between the nodal Moon and the sideral Moon, which may be associated with Moon's anomaly $T_2 = T_1 + \delta T \simeq 27.54$ (the drift of Moon's perigee).

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[7] In Earth's rotating frame the coordinates $\tilde{\mathbf{r}}$ are the same as the coordinates $\mathbf{r}$ in the center-of-mass frame, while velocity is given by $\dot{\tilde{\mathbf{r}}} = \dot{\mathbf{r}} - \mathbf{\Omega} \times \tilde{\mathbf{r}}$. Similar perturbation calculations can be done directly in the rotating frame, with the same results.