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Moon's Problem

(A three-body problem)

An Investigation into "Intractability" and "Uncomputability" in Theoretical Physics

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in Theoretical Physics

Moon's State of the Art

- Numerical solution, directly from Newton's law (Apollo, 1960-1980)
- Accuracy 10^8 (1 meter, distance $\sim 384\,000\text{km}$, velocity $\sim 1\text{km/s}$; Earth's velocity $\sim 30\text{km/s}$)
- All forces, accurate trajectory

Beside computers there are theories

- Theory gives an image, in agreement with empirical observations

Moon's image

- Moon rotates around the Earth (with respect to fixed stars, syderal Moon, 27.32days)
- Moon is transported by Earth (synodal Moon, 29.53days, Moon's phases)
- Moon's trajectory plane oscillates about its inclination with respect to ecliptic (nodal Moon, 27.21days)
- Moon's axis of rotation precesses $\sim 18\text{years}$ (related to nodal Moon)
- Moon's rotation deviates from an ellipse (anomalous Moon, 27.55days, drift of the perigee)

Usefulness of a Theory

- Earth's radius (Eratosthene, by geometrical theory)
- Distances to Moon, Earth, etc (parallax, triangulation, geometry)
- Mass of Earth (Cavendish, by Law of Gravitation)
- Moon's mass, Sun's mass (by motion according to the Law of Gravitation)

Theories

- Newton (1680-1700); Euler, Clairaut, d'Alembert, Laplace, Lagrange, Hamilton-Jacobi, Delauney, Hill (1890); almost all the "mathematical physics"
- Difficulties, as pointed out by Poincare (1900):
 - many small terms in perturbation theory
 - divergencies (asymptotic series, vanishing denominators, resonances)
 - instabilities; "Chaos", instability of the Planetary System?

Cause of Difficulties

- Perturbation to prescribed orbits; periodic orbits
- (As if the problem would be integrable; it is not)
- Actually, the orbit has to be renormalized every step of computations
- Fourier series change frequencies every step
- Consistency of perturbation theory
- (Poincare-Linstedt procedure)

I A New Method for Kepler's Problem

$$E = mr\dot{r}^2/2 + mr^2\dot{\varphi}^2/2 - \alpha/r = mr\dot{r}^2/2 + L^2/2mr^2 - \alpha/r$$

$$L = mr^2\dot{\varphi}$$

(Kepler's second law)

$$U = L^2/2mr^2 - \alpha/r$$

Trajectory: $r_1 = a(1 - e)$ to $r_2 = a(1 + e)$, semi-major axis $a = \alpha/2|E|$

Eccentricity $e = \sqrt{1 - 2L^2|E|/m\alpha^2}$

U_{min} for $r_0 = L^2/m\alpha = a(1 - e^2)$ and $|E| = \frac{\alpha}{2r_0}(1 - e^2)$

Oscillator: Power expansion around the minimum value $r - r_0 = Au$, $A/r_0 = \varepsilon$

$$e^2 = \frac{2\varepsilon^2}{\omega^2}(\dot{u}^2/2 + \omega^2 u^2/2 - \varepsilon\omega^2 u^3 + \dots)$$

$$\omega^2 = \alpha/mr_0^3$$

(Kepler's third law)

Solutions $\varepsilon = e(1 - e)$ (small eccentricities)

$$r = r_0[1 - e \cos \omega t + \frac{e^2}{2}(3 - \cos 2\omega t) + \dots]$$

$$\varphi = \omega t + 2e \sin \omega t - \frac{e^2}{2}(3\omega t - \frac{5}{2} \sin 2\omega t) + \dots$$

$$r = r_0(1 - e \cos \varphi + e^2 \cos^2 \varphi + \dots) = r_0/(1 + e \cos \varphi)$$

Ellipse: semi-major axis $a = r_0/(1 - e^2)$, semi-minor axis $b = r_0/(1 - e^2)^{1/2}$, origin displaced by $ae = r_0e + \dots$ in the focus ae (Kepler's first law)

Technical Note: Resonant (secular) terms, anharmonic corrections

Frequency shift (Poincare-Lindstedt method, 1882-1892)

$$\Omega = \omega(1 - 3e^2/2) = (\alpha/ma^3)^{1/2}$$

(second-order cubic, first-order quartic)

Automatically included by the present method

Other central-force fields

$$r = r_0[1 - e \cos \omega t + \frac{\beta e^2}{2}(3 - \cos 2\omega t) + \dots]$$

$$\varphi = \sqrt{v_1/(3v_1 + r_0v_2)}\{\omega t + 2e \sin \omega t - \frac{e^2}{2}[3(2\beta - 1)\omega t - \frac{2\beta + 3}{2} \sin 2\omega t] + \dots\}$$

$$m\omega^2 = 3v_1/r_0 + v_2, \beta = (2v_1 - r_0^2v_3/6)/(3v_1 + r_0v_2)$$

$$r = r'_0[1 - e \cos \chi + (2 - \beta)e^2 \cos^2 \chi + \dots], \quad r'_0 = r_0[1 - 2(1 - \beta)e^2]$$

$$\varphi = \sqrt{v_1/(3v_1 + r_0v_2)}\chi$$

Closed orbits. First sign of "chaos"

$$\sqrt{v_1/(3v_1 + r_0v_2)} = p/q$$

Gravitational potential: $\beta = 1, p/q = 1$

Spatial oscillator ($v(r) = const + \alpha r^2$): $\beta = 1/2, p/q = 1/2, \chi = 2\varphi$ (ellipse centered on the origin)

Only two potentials close exactly the orbits (Bertrand's theorem, 1873)

$$p/q = (1/\pi) \int_{r_1}^{r_2} dr \cdot (L/r^2) / \sqrt{2m(E - v) - L^2/r^2}$$

(closure integral)

Many other fields may close the orbits to any finite-order of perturbation theory

But not in the limit

To the extent to which an irrational number is approximated by rational numbers

Infinitely (and densely) quasi-closed orbits: first sign of "chaos"

- Sensitive and arbitrary dependence on initial conditions (L, E)
- Compute closure integral; would never know whether it is rational or irrational
- Unprovability, Undecidability

II Moon's problem: a three-body problem

$$E = m_1 \dot{\mathbf{r}}_1^2/2 + m_2 \dot{\mathbf{r}}_2^2/2 - Gm_0m_1/r_1 - Gm_0m_2/r_2 - Gm_1m_2/|\mathbf{r}_1 - \mathbf{r}_2|$$

$m_0 \simeq 2 \times 10^{30} Kg$ (Sun), $m_1 \simeq 6 \times 10^{24} Kg$ (Earth), $m_2 \simeq 7 \times 10^{22} Kg$ (Moon)

$G \simeq 6.7 \times 10^{-11} m^3/Kg \cdot s^2$

$r_1 \simeq 150 \times 10^6 Km$ (Sun-Earth), $r_2 \simeq 384\,000 Km$ (Earth-Moon)

$$\mathbf{L}_{tot} = m_1 \mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2 \mathbf{r}_2 \times \dot{\mathbf{r}}_2$$

Center of mass $E = M\dot{\mathbf{R}}^2/2 + m\dot{\mathbf{r}}^2/2 -$

$$-Gm_0m_1/|\mathbf{R} - m_2\mathbf{r}/M| - Gm_0m_2/|\mathbf{R} + m_1\mathbf{r}/M| - Gm_1m_2/r$$

Quadrupolar perturbation: $r/R \sim 3 \times 10^{-3} ((r/R)^2 \sim 10^{-5})$

$$E = M\dot{\mathbf{R}}^2/2 + m\dot{\mathbf{r}}^2/2 - \alpha/R - \beta/r - \gamma[3(\mathbf{r}\mathbf{R})^2/R^2 - r^2]/R^3$$

$$\alpha = Gm_0M, \beta = GmM \text{ and } \gamma = Gm_0m/2$$

Two coupled Kepler's problems $E = E_1 + E_2 + \gamma v$

$$E_1 = M\dot{\mathbf{R}}^2/2 - \alpha/R, \quad E_2 = m\dot{\mathbf{r}}^2/2 - \beta/r, \quad v = -r^2(3 \cos^2 \chi - 1)/R^3$$

Another three-body problem: Jupiter-Saturn Couple

$$E = m_1\dot{\mathbf{r}}_1^2/2 + m_2\dot{\mathbf{r}}_2^2/2 - Gm_0m_1/r_1 - Gm_0m_2/r_2 - Gm_1m_2/|\mathbf{r}_1 - \mathbf{r}_2|$$

Perturbation:

$$Gm_1m_2/|\mathbf{r}_1 - \mathbf{r}_2|$$

$m_1, m_2 \ll m_0$ and \mathbf{r}_1 not too close to \mathbf{r}_2

The "Four Moons": four periodicities; the Greeks, the Babylonians,...

Sideral year: $T_0 = 365.26$ days

Sideral Moon: $T_1 = 27.32$ days (rotation about fixed stars)

Synodal Moon: ($T_4 = 29.53$ days) (combining the year and the sideral Moon; Moon's phases)

Nodal Moon: $T_3 = 27.21$ days (up and down about the ecliptic; rotation of the axis ~ 18 years)

Anomalous Moon: $T_2 = 27.55$ days (deviation from ellipse, the drift of the perigee)

"Second-of-time" accuracy: five decimals; The Greeks. the Babylonians, ... !

Polar coordinates:

$$E = E_1 + E_2 + \gamma v$$

$$E_1 = M\dot{R}^2/2 + MR^2(\dot{\Theta}^2 + \dot{\Phi}^2 \sin^2 \Theta)/2 - \alpha/R$$

$$E_2 = m\dot{r}^2/2 + mr^2(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta)/2 - \beta/r$$

$$v = -r^2(3 \cos^2 \chi - 1)/R^3, \quad \cos \chi = \sin \Theta \sin \theta \cos(\Phi - \varphi) + \cos \Theta \cos \theta$$

First order of the perturbation theory

Eccentricity series: $R_0 = L^{(0)2}/M\alpha$, $\Omega^2 = \alpha/MR_0^3$ (Earth period ~ 365 days) and $e_1 = (1 - 2R_0 |E_1|/\alpha)^{1/2}$ (Earth eccentricity $e_1 = 0.017$)

$$R^{(0)} = R_0[1 - e_1 \cos \Omega t + \frac{e_1^2}{2}(3 - \cos 2\Omega t) + \dots]$$

$$\Phi^{(0)} = \Omega t + 2e_1 \sin \Omega t - \frac{e_1^2}{2}(3\Omega t - \frac{5}{2} \sin 2\Omega t) \dots]$$

$$\Theta^{(0)} = \pi/2$$

Relative motion: $r_0 = l^{(0)2}/m\beta$, $\omega^2 = \beta/mr_0^3$ ($\omega \gg \Omega$; Moon's period ~ 27 days), eccentricity $e_2 = (1 - 2r_0 |E_2|/\beta)^{1/2}$ (Moon's orbit eccentricity $e_2 \simeq 0.055$)

Tilted frame:

$$\varphi' = \varphi - \frac{1}{4}\theta_0^2 \sin 2\varphi + \dots = \omega t + 2e_2 \sin \omega t - \frac{1}{4}\theta_0^2 \sin 2\omega t + \dots$$

$$\theta' = \pi/2 - \theta_0 \sin \varphi + \dots = \pi/2 - \theta_0 \sin \omega t + \dots$$

$$r' = r = r_0(1 - e_2 \cos \omega t + \dots)$$

Another perturbation parameter: tilting angle

$$\theta_0 = 5^\circ = \pi/36$$

Perturbation theory: Six coordinates $R, \Theta, \Phi, r, \theta, \varphi$

Triple series expansion: eccentricities ($e_{1,2}$), inclination (θ_0) and γ - interaction (Weierstrass, Sweden King's contest)

Zeroth order: $\gamma v/(\alpha/R) \sim 10^{-7}$, independent Kepler's problem for the Earth

$$\gamma v/(\beta/r) \sim 10^{-3}, r', \theta', \varphi' \text{ and first-order corrections}$$

$$r = r' + \gamma r_1 + \dots, \theta = \theta' + \gamma \theta_1 + \dots, \varphi = \varphi' + \gamma \varphi_1 + \dots$$

Zeroth order in interaction: External field

$$R^{(0)} = R_0, \Theta^{(0)} = \pi/2, \Phi^{(0)} = \Omega t, r^{(0)} = r_0, \theta^{(0)} \simeq \pi/2, \varphi^{(0)} = \omega t$$

Results:

$$r_1 = -(2r_0^4/\beta R_0^3) \cos 2\omega t, \varphi_1 = -(5r_0^3/4\beta R_0^3) \sin 2\omega t$$

Frequency shift $\omega' = \omega(1 - \gamma r_0^3/2\beta R_0^3) = \omega(1 - \Omega^2/4\omega^2)$

$$\Omega/\omega \simeq 1/13$$

Newton (Euler, Clairaut, d'Alembert, Laplace, Lagrange, Jacobi, Delauney)

ω' and γ -correction $\delta\omega/\omega = -\Omega^2/4\omega^2$, syderal Moon T_1 (27.32days)

Rotating frame $\omega' - \Omega$, synodal Moon T_4 (29.53days)

Correction: z -oscillation, $\tilde{\omega}^2 = \omega^2 + \Omega^2 = \omega'^2(1 + \Omega^2/2\omega^2) + \Omega^2$ (rotating frame)

Newton's correction: $\delta\omega/\omega' = 3\Omega^2/4\omega^2$, nodal Moon T_3 (27.21days)

Frequency $\delta\omega$ leads to $4\omega/3\Omega^2 \simeq 18$ years of Moon's orbit plane retrograde motion

Anomalous Moon: correction $3\Omega^2/2\omega^2 \sim 0.008$ applied to ω' (27.55 days; factor 2; ~ 1.7 by d'Alembert and Clairaut, ~ 1750)

Restricted Perturbations

Delauney ~ 1860 up to ~ 500 terms! (~ 2000 print pages)

Hill ~ 1880 (rotating frame)

Poincare ~ 1890 ; Sweden's King contest; stability of the Planetary System

Weierstrass and Sundman

Poincare-Mittag-Leffler 4 years mistake ~ 1900

Gutzwiller et al ~ 1980 ; quasi-failure; **Slow convergence, weak accuracy, algorithms**

Modern computers: 1960-1980 (aselenization)

Poincare and the "weak" chaos

4 (apparent) fundamental frequencies (Ω, ω', γ -corrections); rotating frame

Solution looks quite erratic (main pattern plus infinitely small, fine laceworking!)

Poincare "Weak" Chaos

Small contributions are important: **a small error on Earth, a big failure on Moon!**

(1km on Moon, one second of time, $2''$, 10^5 accuracy!)

Missing integrals, instabilities, the "strong" chaos

6 degrees of freedom in 3-body problem and only 4 integrals (E, \mathbf{L}_{tot})

Are there other integrals?

No !(Bruns-Poincare theorem) (algebraic in parameters)

Non-anayticity: $\dot{\varphi}$ very high over the pole ($\theta = 0$), sudden change of the trajectory along a longitudinal circle

Instabilities, "strong" chaos

Rather strange external perturbations

Not only Chaos (weak or strong)

In addition:

- a great deal of proliferating contributions
- vanishing denominators, resonances, divergencies
- time-consuming just for keeping track of them (parallel computing?)
- more computing time than the real, natural process (more program bites than number of bits produced)
- "Uncomputability", "Computational Irreducibility", "Intractability"

III A new route to quantizing the Hydrogen atom

A special classical motion: passing through the centre

$$E = m\dot{r}^2/2 + L^2/2mr^2 - \alpha/r$$

$$L = 0, E = -\alpha/r_0$$

Oscillations between 0 and r_0 around $r_0/2$

$$\dot{u}^2 + \omega^2(u-1)/(u+1) = 0, r = r_0(1+u)/2, \omega^2 = 8\alpha/mr_0^3 = 8|E|^3/m\alpha^2$$

Solution $2 \arcsin \sqrt{(1-u)/2} + \sqrt{1-u^2} = \omega t$, $-1 < u < 1$, periodicity, $\omega/2$

Quantization

$$m\dot{\rho}^2/2 + m\omega^2\rho^2/2 + \dots - \alpha/r_0 = 0$$

Harmonic oscillator

$$\hbar\omega(n+1/2)/2 = \alpha/r_0, \hbar\omega\delta n/2 = |E|_q, \delta n = n$$

$$|E|_q = \frac{m\alpha^2}{2\hbar^2 n^2}$$

Anharmonic corrections, frequency shift, renormalization of energy, etc

Case $L \neq 0$

$$r_0 = L^2/m\alpha, m\omega^2 = \alpha/r_0^3$$

$$E = m\dot{r}^2/2 + (\alpha/2r_0^3)(r - r_0)^2 + \dots - \alpha/2r_0, \quad \hbar\omega(n + 1/2)/2 = \alpha/2r_0 + E$$

$$L^2/2I = L^2/2mr_0^2 = \alpha/2r_0 = \text{Energy}$$

$$\hbar\omega(\text{Energy})\delta n/2 = \text{Energy}$$

$$\text{Energy} = \frac{m\alpha^2}{2\hbar^2 n^2}$$

Closed orbits, exact formula

A general central-field potential $v(r)$

$$-L^2/mr_0^3 + v_1 = 0, \quad \text{Energy} = L^2/2mr_0^2 = v_1 r_0/2$$

$$r_0(\text{Energy})$$

$$\omega^2 = 3v_1/mr_0 + v_2/m$$

$$\hbar\omega(\text{Energy})\delta n/2 = \text{Energy}$$

$$\hbar^2[3v_1(\text{Energy})/mr_0(\text{Energy}) + v_2(\text{Energy})/m]n^2/4 = \text{Energy}^2$$

Approximate, open orbits

IV A few comments on Chaos

- 1 Endless orbits never repeating
- 2 Sensitive dependence on initial conditions (Lyapunov exponents)
- 3 Non-linearities
- 4 Logistic maps, bifurcation and Feigenbaum number (~ 4.67 ; $x_{n+1} = Lx_n(a - x_n)$; ratio of two successive L's)
- 5 Fractal dimension ($r = 3^n$, $N = 4^n$: $N = r^D$, $D = \ln 4 / \ln 3 = 1.26$; Koch's curve)
- 6 Cellular automata

What we see?

Chaos (open orbits, great variability in small things with huge consequences)

Unprovability, undecidability (closing orbits)

Intractability, uncomputability and computational irreducibility (3-body problem)

Universality (Nature is a Universal Machine, emulate everything)

Questions

- 1 What about friction? (Earth, loses 6 minutes per century)
- 2 What about the effect of the other Planets on Moon's motion? (time-space stochastic perturbation)?
- 3 What about the couple Jupiter-Saturn? (Laplace 2 : 5 resonance, Jupiter 12 years, Saturn 30 years)?
- 4 Another resonance, Moon's ever-staring side (Lagrange, distorted Moon, the coupling, pin-down the motion; Gauss and the blocking of the phase)
- 5 The great analogy with CDW's (or SDW's); pinning, commensurate, incommensurate, order parameter, symmetry breaking, Goldstone modes, etc, etc
- 6 Dipolar (and multipolar) coupling
- 7 What about relativistic corrections? ($v/c \sim 10^{-5}$)
- 8 Zeeman effect for Hydrogen atom, semiclassical orbits, quantization

.....

A New Research Program

Investigations into "Intractability" and "Uncomputability" of the 3-Body Problem