Journal of Theoretical Physics

Founded and Edited by M. Apostol **129** (2006)

ISSN 1453-4428

Elastic scattering in short-range potentials

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Abstract

The scattering amplitude for elastic scattering in short-range potentials is obtained by means of an approximate solution of the integral Schrodinger equation. The solution is exact for δ-type potentials. In the long wavelength limit the scattering amplitude is the one given by the effective-range theory (s-wave), the scattering length and the parameter of the effective range being given in terms of the potential depth and range for potentials that are not very strong. It has a non-perturbational character. In the short wavelength limit it reproduces closely the quasi-classical theory. In the intermediate range of wavelengths the main contribution comes from Born's scattering amplitude, which must be corrected in order to satisfy the optical theorem.

The scattering is adequately described by the phase-shift theory. However, it requires the computations of the phase shifts, which amounts, more or less, to solving the Schrodinger's equation. We present here a method of obtaining an approximate solution of the integral Schrodinger equation (equation of the potential) for short-range potentials, which provides a satisfactory representation for the scattering amplitude.

Schrodinger's equation for potential $V(\mathbf{r})$ reads

$$(\Delta + k^2)\psi = 4\pi U(\mathbf{r})\psi \ , \tag{1}$$

where $k = \sqrt{2mE/\hbar^2}$ is the wavevector of the ingoing particle with energy E and (reduced) mass m, and $U(\mathbf{r}) = mV(\mathbf{r})/2\pi\hbar^2$. We look for a scattering solution of this equation of the form $\psi = \exp(ikz) + \varphi(r)$, where $\varphi(r) = f \cdot \exp(ikr)/r$ behaves like an outgoing wave at infinity. Function f is the scattering amplitide. The integral representation of equation (1) reads

$$\varphi(\mathbf{r}) = -\int d\mathbf{r}' \cdot U(\mathbf{r}')[e^{ikz'} + \varphi(\mathbf{r}')]\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \ . \tag{2}$$

The short-range potential is vanishing beyond a certain range b. Then, for $r \gg b$, we can disentangle the $\mathbf{r}$-dependence from the $\mathbf{r}'$-dependence in equation (2), by writing $k\,|r - r'| \simeq kr - \mathbf{k}'\mathbf{r}'$, where $\mathbf{k}' = \mathbf{k}r/r$ is the outgoing wavevector. We get the asymptotic form of the solution

$$\psi(\mathbf{r} \sim \infty) = -[U(\mathbf{q}) + \int d\mathbf{r}' \cdot U(\mathbf{r}')\varphi(\mathbf{r}')e^{-i\mathbf{k}'\mathbf{r}'}]\frac{e^{ikr}}{r} \ , \tag{3}$$

where

$$U(\mathbf{q}) = \int d\mathbf{r} \cdot U(\mathbf{r})e^{-i\mathbf{q}\mathbf{r}} \tag{4}$$

is the well-known Born's scattering amplitude with the wavevector transfer $\mathbf{q} = \mathbf{k}' - \mathbf{k}$, $q^2 = 2k^2(1-\cos\theta)$. In order to get the full solution we need also the wavefunction $\varphi(\mathbf{r} \sim 0)$ for small values of $\mathbf{r}$, *i.e.* comparable with the range of the potential. To this end, we write equation (2) approximatelly as

$$b'\varphi(\mathbf{r} \sim 0) \simeq -U(\mathbf{k}) - \int d\mathbf{r}' \cdot U(\mathbf{r}')\varphi(\mathbf{r}') \tag{5}$$

where $b' = be^{-ikb} \simeq b(1-ikb)$; b'^{-1} is an order-of-magnitude estimation for the factor $e^{ik|\mathbf{r}-\mathbf{r}'|}/\left|\mathbf{r}-\mathbf{r}'\right|$. According to equation (5), the wavefuncton φ does not depend on $\mathbf{r}$. This approximation is valid for potentials that are sufficiently weak, as, for instance, potential wells having one bound state at most. It is a straightforward matter to solve equation (5) and to get, by (3), the scattering amplitude

$$f = -U(\mathbf{q}) + \frac{U(\mathbf{k})U(\mathbf{k}')}{b' + U(\mathbf{k}=0)} \ . \tag{6}$$

It is easy to see that for a δ-type potential $U(\mathbf{r}) = U_0\delta(\mathbf{r})$ equation (2) has an exact solution $f = (-U_0^{-1} - b^{-1} - ik)^{-1}$ which coincides with (6). It satisfies the optical theorem. The δ-potential is viewed as $b \to 0$ and $V \to \infty$, such that $U_0 = mVb^3/2\pi\hbar^2 = const.$

In the long wavelengths limit $kb \ll 1$ the Fourier transform of the potential in equation 6) has a weak k-dependence. For central potentials we can write $U(\mathbf{k}) = U_0 - Ak^2$, where $U_0 = U(\mathbf{k}=0)$ and $A > 0$. Limiting ourselves to s-wave we get from (6)

$$f_0 = \frac{1}{-a^{-1} + r_0k^2/2 - ik} \ , \tag{7}$$

where $a = bU_0/(b+U_0)$ is the scattering length and $r_0 = -4A(b+U_0)/bU_0^2$ is the parameter of the effective-range theory. For $U_0 \sim -b$ this is Wigner's formula for resonance. For U_0 slightly below $-b$, there may exist a shallow bound state with energy $\varepsilon < 0$ and radius $R = a = 1/\kappa$, where $\kappa^2 = 2m\left|\varepsilon\right|/\hbar^2$. The scattering length and the parameter r_0 are positive in this case. For U_0 slightly above $-b$ there may exist a virtual state, and both a and r_0 change sign. We note that near resonance the coefficient of the p-wave in the expansion of Born's amplitude is a second-order quantity in powers of $(b+U_0)$ ($\sim (b+U_0)^2/b^2U_0^2$). It is worth noting the non-perturbational character of solution (6) in s-wave.

In the intermediate range of wavelengths $kb \sim 1$ the main contribution to equation (6) comes from Born's amplitude. For central potentials, its partial waves U_l must be corrected by $U_l \to U_l/(1-ikU_l)$ in order to satisfy the optical theorem. This correction is small, of the order of the second-term left aside in equation (6).

Toward short wavelengths limit $kb \gg 1$ equation (6) begins to assume a more perturbational character, as expected. In this limit the Born amplitude given by (4) vanishes unless the wavevector transfer $\mathbf{q}$ is small and perpendicular to $\mathbf{k}$, such that $q \simeq k\theta$. For central potentials, Born's amplitude can then be written as

$$\begin{aligned} U(\mathbf{q}) &= \int d\mathbf{r} \cdot U(\mathbf{r})e^{-i\mathbf{qr}} = \int d\rho \cdot [\int_{-\infty}^{+\infty} dz \cdot U(z,\rho)]e^{-i\mathbf{q}\rho} = \\ &= \tfrac{4\pi}{k^2(\Delta\theta)^2} \int_{-\infty}^{+\infty} dz \cdot U(z,0) \ , \end{aligned} \tag{8}$$

where ρ is the transverse position vector with respect to wavevector $\mathbf{k}$ and $\Delta\theta$ is the range of angles around $\theta = 0$ where the amplitude is non-vanishing. Similarly, the following set of successive approximations

$$\begin{aligned} U(\mathbf{k}) &= \int d\mathbf{r} \cdot U(\mathbf{r})e^{-i\mathbf{kr}} = \int d\rho dz \cdot U(z,\rho)e^{-ikz} = \\ &= \pi b^2 \int dz \cdot U(z,0)e^{-ikz} = (\pi b^2/bk) \int dz \cdot U(z,0) \end{aligned} \tag{9}$$

can be used for $U(\mathbf{k})$, and

$$U(\mathbf{k}') = \int d\rho dz \cdot U(z,\rho) e^{-ikz} e^{-i\mathbf{q}\rho} = [4\pi/k^2(\Delta\theta)^2](1/bk) \int dz \cdot U(z,0) \tag{10}$$

for $U(\mathbf{k}')$. In equation (6) we may neglect now $U(\mathbf{k}=0)$ in the denominator, and get

$$f = \frac{1}{-k^2(\Delta\theta)^2/4\pi A - ik(\Delta\theta)^2/4} \ , \tag{11}$$

where $A = \int dz \cdot U(z,0)$. The partial-wave amplitudes are readily obtained from (11) as

$$f_l = \frac{1}{-k^2/\pi A - ik} \ . \tag{12}$$

These results coincide with those of the quasi-classical theory of high-energy scattering in the second-order approximation.

In conclusion, one may say that the approximate solution (6) of the integral Schrodinger equation for elastic scattering in short-range potentials provides a fair, and unifying representation for rather disparate results like the non-perturbational theory of effective range in s-wave and the quasi-classical theory of high-energy scattering.