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On a non-linear diffusion equation describing clouds and wreaths of smoke

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Abstract

A non-linear diffusion equation is derived, by taking into account the local variations of solvent pressure or volume, within a mechanism of diffusion driven by collisions between the fluid particles. A particular class of radially-symmetric solutions is discussed, which are localized over finite ranges, centered either on the origin or on finite values of the radius. In the case of a thin layer of solute they look like disks or rings, and we suggest that they might be appropriate for describing the shapes of clouds or wreaths of smoke. These patterns exhibit diffusion fronts propagating slower and slower, and grow in time at a constant rate of the total number of solute particles.

Recently,[1] the ascent of warm, moist air in the Earth's atmosphere has been modelled with the Kardar-Parisi-Zhang equation,[2] known from the crystal growth on atomic surfaces. This equation has been solved numerically,[3],[4] the fractal aspects of its solutions being emphasized. Leaving aside the randomness term responsible for the fractal behaviour, the remaining part of this equation looks like a non-linear diffusion equation, though the non-linear term has originally been interpreted as describing growth. By taking into account the local variations of the solvent pressure within a diffusion mechanism driven by the collisions between the fluid particles we show here that this non-linear term may indeed arise in a diffusion process in non-equilibrium conditions; while at equilibrium, the local variations of the solvent volume lead to an identical term, only with opposite sign. The non-linear diffusion equation obtained thereby is solved for a particular class of radially-symmetric solutions. In the first case it is found that the solute concentrations extend over finite ranges in space, having the shape of a disk in two dimensions. In the second case solutions of annular shape are obtained in two dimensions. These patterns of the solute concentration exhibit diffusion fronts propagating slower and slower, and grow in time at a constant rate of the total number of solute particles. We suggest that they might be interpreted as describing clouds or wreaths of smoke. Specifically, we discuss the two-dimensional case, but relevant features, wherever appropriate, are pointed out in one and three dimensions.

Suppose that we have a thin layer of fluid and that its particles may jump from the position (x, y) at time t to the neighbouring positions $(x \pm a, y)$, $(x, y \pm a)$, where a is an average, atomic length scale; and suppose further on that these jumps proceed with an average frequency $\nu(x, y)$. Then, the time variation of the fluid concentration $n(x, y, t)$ is given by[5]

$$\begin{array}{c}\frac{\partial n}{\partial t} = \nu(x+a, y)n(x+a, y) + \nu(x-a, y)n(x-a, y) - 2\nu n + \\ \nu(x, y+a)n(x, y+a) + \nu(x, y-a)n(x, y-a) - 2\nu n \quad .\end{array} \tag{1}$$

Here we have assumed that n varies slowly in time over lapses of time much longer than the time scale ν^{-1}; and we assume further on that n and ν vary slowly in space too, over distances much larger than a. Then, by series expansion (1) becomes

$$\frac{\partial n}{\partial t} = a^2\nu\left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2}\right) + 2a^2\left(\frac{\partial \nu}{\partial x}\frac{\partial n}{\partial x} + \frac{\partial \nu}{\partial y}\frac{\partial n}{\partial y}\right) + a^2 n\left(\frac{\partial^2 \nu}{\partial x^2} + \frac{\partial^2 \nu}{\partial y^2}\right) . \tag{2}$$

We are interested in small values of the solute average concentration n, so that we may neglect the last term in (2); we obtain therefore

$$\frac{\partial n}{\partial t} = a^2\nu \triangle n + 2a^2 grad\nu \cdot gradn \ . \tag{3}$$

We adopt now a kinetic model of fluid, wherein the collision frequency ν is given by $\nu = (1/3)\,\sigma u N$, σ being the particle cross-section, u- the particle mean velocity and N- the solvent concentration. We assume that by adding n solute particles per unit volume (of similar nature as those of the solvent) the non-equilibrium local pressure of the solvent changes from $p = (1/3)\,mu^2 N$ to $p^{'} = p/(1-n/N) = (1/3)mu^{'2}N$, where m is the mass of the fluid particles; whence, for $n/N \ll 1$, we obtain $\nu^{'} = b(N+n/2)$, with $b = (1/3)\sigma u$. Similarly, when equilibrium is established by local variations of the solvent volume $p^{'} = p/(1+n/N)$ and $\nu^{'} = b(N-n/2)$. The first process is expected to occur more frequently in dilute gases, while the latter is more appropriate for dense gases or liquids. Introducing these n-dependences of the collision frequency ν in (3) we get in the former case

$$\frac{1}{S}\frac{\partial n}{\partial t} = \triangle n + A(gradn)^2 \ , \tag{4}$$

and

$$\frac{1}{S}\frac{\partial n}{\partial t} = \triangle n - A(gradn)^2 \ , \tag{5}$$

in the latter, where $S = a^2 bN$ and $A = 1/N$. These are non-linear diffusion equations, the first one (equation (4)) being the non-stochastic part of the Kardar-Parisi-Zhang equation.[2] We note that the non-linear term in (4) and (5) occurs as a result of taking into account the local variations of the solvent pressure or volume within a mechanism of diffusion driven by the collisions between the fluid particles. This is in contrast with the diffusion by tunneling in an external potential, as, for example, in a (homogeneous) solid, where the jump frequency ν does not depend on the solute concentration n. Of course, if N is very large then we may neglect the non-linear term in (4) and (5), and we get the usual, linear diffusion equation.

We note, first, that the non-linear diffusion equation given above by (4) and (5) does not conserve the total number of solute particles, in contrast with the linear diffusion equation. Indeed, we have

$$\frac{\partial}{\partial t}\int n \cdot d\mathbf{r} = \pm SA\int (gradn)^2 \cdot d\mathbf{r} \ , \tag{6}$$

and requiring the conservation of the particle number would imply the trivial solution $n = const$. We shall look for radially-symmetric solutions (as those corresponding to radially-symmetric initial conditions), in which case (4) reads

$$\frac{1}{S}\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial r^2} + \frac{1}{r}\frac{\partial n}{\partial r} + A\left(\frac{\partial n}{\partial r}\right)^2 \ ; \tag{7}$$

further on, we assume that n depends on $\xi = r/\sqrt{t}$ only, which represents a particular class of solutions, corresponding to an increase of the total number of solute particles at a constant rate,

$$\int n(\xi) \cdot d\mathbf{r} = const \cdot t \tag{8}$$

($\sqrt{t}$ in one dimension, $t^{3/2}$ in three dimensions). Equation (7) reads then

$$n^{''} + \left(\frac{1}{\xi} + \frac{\xi}{2S}\right) n^{'} + An^{'2} = 0 \ , \tag{9}$$

which is a Bernoulli-type equation. Introducing $f = 1/n^{'}$ we get an equation of the type

$$f^{'} + af + b = 0 \ , \tag{10}$$

whose solution is[6]

$$f = e^{-F}\left(const - \int^{\xi} be^{F} \cdot d\xi\right) \ , \tag{11}$$

where $F = \int^{\xi} a \cdot d\xi$. In this way we obtain for (9)

$$n^{'} = \frac{\exp\left(-\xi^2/4S\right)}{\xi\left[const + \frac{1}{2}A \cdot \mathrm{Ei}\left(-\xi^2/4S\right)\right]} \ , \tag{12}$$

where Ei is the exponential integral ($\mathrm{Ei}(x) = -\int_{-x}^{\infty}\left(e^{-t}/t\right) \cdot dt$). The exponential integral $\mathrm{Ei}\left(-\xi^2/4S\right)$ is a monotonously increasing function which has the following asymptotic behaviours:

$$\begin{aligned} \mathrm{Ei}\left(-\xi^2/4S\right) &\sim \ln\left(\xi^2/4S\right) \ , \quad \xi^2/4S \ll 1 \ , \\ \mathrm{Ei}\left(-\xi^2/4S\right) &\sim -\tfrac{1}{(\xi^2/4S)} \cdot e^{-\xi^2/4S} \ , \quad \xi^2/4S \gg 1 \ . \end{aligned} \tag{13}$$

It is easily now to see that for $const \geq 0$ in (12) we would get a negative, unphysical solution n, while for $const < 0$ we get a monotonously decreasing solution n whose asymptotics are

$$n \sim \frac{1}{A} \ln\left|\ln\left(\xi^2/4S\right)\right| \ , \quad \xi^2/4S \ll 1 \ , \tag{14}$$

$$n \sim \frac{1}{\xi^2/4S} \cdot e^{-\xi^2/4S} \ , \quad \xi^2/4S \gg 1 \ . \tag{15}$$

The solute small density is therefore given by the smooth function n given by (15) for $\xi^2/4S \gg 1$, while it increases rapidly for $\xi^2/4S \ll 1$, as suggested by (14). There exists, consequently, a front of diffusion placed at $\xi^2/4S \sim 1$, *i.e.* at $r \sim 2\sqrt{St}$, which propagates slower and slower ($dr/dt \sim 1/\sqrt{t} \longrightarrow 0$ for $t \longrightarrow \infty$), encompassing an area which increases linearly with time, exactly as the total number of solute particles does. We may say that the solute particles placed initially over a certain, small area, propagate rather compactly (with a diffusion front), slower and slower, while fed continuously at a constant rate. This suggests that we may speak in this case of a disk-like, cloud pattern, as an atmospheric, or a smoke cloud.

We note that a similar solution exists also for the linear diffusion equation ($A = 0$), $n = const \cdot \mathrm{Ei}(-\xi^2/4S)$, but not for the number-conserving solution of the linear diffusion equation

$$n \sim \frac{1}{t} \cdot \exp(-\xi^2/4S) \ , \quad \xi^2/4S \gg 1 \ , \tag{16}$$

which decreases uniformly to zero for $t \longrightarrow \infty$.

Similar solutions exist for the non-linear equation in three dimensions and in one dimension, but in the later case the existence of the diffusion front is doubtful, since the solution n and its derivative are finite at the origin.

A similar analysis can be carried out straightforwardly for (5), which amounts to changing the sign of A in (9) and (12). It is easily to see, in this case, that we would get an unphysical solution for *const* ≥ 0 in (12), but a physical one for *const* < 0 (and $\xi^2/4S \gg 1$). However, this solution has a logarithmic singularity at certain, finite values ξ_0 (depending on this integration constant), where it looks like $n \sim -\ln\left|(\xi-\xi_0)/2\sqrt{S}\right|$. For large values of the variable $\xi^2/4S$ the solution behaves like the previous one, given by (15). One may say, therefore, that there exists in this case a finite value ξ_0 (and a finite value of the radius r_0), where the solute density is concentrated; it extends over a finite range of the order ξ_0, so that we may identify two diffusion fronts, one by each side of the singularity, propagating (in opposite directions) slower and slower, and encompassing an area which increases again linearly in time, in the same manner as the total number of solute particles. The pattern has an annular shape, suggesting again atmospheric clouds or wreaths of smoke. We remark that such a solution might be expected for the equilibrium diffusion process, corresponding to (5). Indeed, a spot of solute particles cannot proceed to diffuse at equilibrium by removing the solvent particles only outwardly, they have to displace them inwardly too, acquiring thereby the ring-like shape of an annulus.

Similar conclusions may also be reached in one and three dimensions, though in the former case the solution may be peaked at the origin, too ($\xi_0 = 0$).

Finally, we mention that the smoothness conditions required for deriving (3) are met by our solutions given by (15) for distances $\triangle r$ and lapses of time $\triangle t$ much smaller than the distance r and, respectively, the time t of observation (but certainly much larger than a and, respectively, ν^{-1}); we note also that a low diffusion coefficient S favours the fulfillment of the asymptotic condition $\xi^2/4S \gg 1$ where our solutions hold.

References

[1] J. D. Pelletier, e-print 9605001, ao-sci@xxx.lanl.gov

[2] M. Kardar, G. Parisi and Y. C. Zhang, Phys. Rev. Lett **56** 889 (1986).

[3] J. G. Amar and F. Family, Phys. Rev. **A41** 3399 (1990).

[4] K. Moser, D. E. Wolf and J. Kertesz, Physica **A178** 215 (1991).

[5] F. Despa and M. Apostol, Solid State Commun. **94** 153 (1995); J. Phys.&Chem. Solids, to appear.

[6] See, for example, E. Kamke, *Differential Equations,* Lepzig (1959).