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Coulomb scattering

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Schrodinger's equation for Coulomb potential $V = \alpha/r$ reads

$$(\Delta - \frac{2}{Dr})\psi = -k^2\psi \ , \tag{1}$$

where $D = \hbar^2/m\alpha$ is the Coulomb localization length ("Bohr" radius) and $k^2 = 2mE/\hbar^2 > 0$ is the wavevector of the energy E for the scattering of a particle of mass m.

According to the definition of D, the localization energy $\hbar^2/mD^2$ compares with Coulomb energy α/D, so that the Coulomb potential can be viewed as a perturbation for large D, *i.e.* for $kD = \hbar v/\alpha \gg 1$, where v is particle's velocity. It means $\alpha \ll \hbar^2 k/m = \hbar v$ (the opposite limit is the quasi-classical limit).

Writing $r = \sqrt{\rho^2 + z^2}$, one can see that ψ is singular for $\rho \rightarrow 0$. This happens either for $z = r$ or for $z = -r$. We choose $z = r$, corresponding to an incoming plane wave e^{ikz} for $z \rightarrow -\infty$. Equation (1) becomes asymptotically

$$d^2\psi/dz^2 - (2/Dr)\psi = -k^2\psi \ , \tag{2}$$

whose solution is

$$\psi = e^{ikz+(i/kD)\ln k(r-z)} \ . \tag{3}$$

One can see that the plane wave is distorted by forward scattering because of the long range of the Coulomb potential. In addition, the distortion is vanishing for $kD \gg 1$ for any finite scattering angle. A series expansion in powers of $1/D$ corrects also the amplitude of the plane wave, not only its phase.

Leaving aside the centrifugal potential, the spherical wave $\psi = \chi/r$ satisfies

$$\chi'' - (2/Dr)\chi = -k^2 \ ; \tag{4}$$

in compliance with the scattering boundary condition, we write $r \rightarrow A - z$, where $A = const \rightarrow r$. The wavefunction becomes $\psi = \chi/(A - z)$ and equation (4) reads now

$$d^2\chi/dz^2 - (2/Dr)\chi = -k^2\chi \ ; \tag{5}$$

its solution is

$$\chi = e^{ik(A-z)-(i/kD)\ln(r-z)} \rightarrow e^{ikr-(i/kD)\ln(r-z)} \ . \tag{6}$$

Similarly,

$$\psi = \frac{1}{A-z}e^{ikr-(i/kD)\ln(r-z)} \rightarrow \frac{1}{r-z}e^{ikr-(i/kD)\ln(r-z)} \ . \tag{7}$$

The full scattering solution reads then

$$\psi = e^{ikz+(i/kD)\ln k(r-z)} + f \cdot e^{ikr-(i/kD)\ln(2kr)}/r \ , \qquad (8)$$

where the scattering amplitude is given by

$$f = -\frac{1}{2k^2 D \sin^2 \theta/2} e^{-(2i/kD)\ln(\sin\theta/2)} \ . \qquad (9)$$

The prefactor in (8) is chosen by dimensional arguments, such as to agree with the $1/D$-expansion. Additional phase factors depending on k are also present in the scattering amplitude.

Equation (9) gives the Rutherford cross-section $d\sigma = 1/4k^4 D^2 \sin^4 \theta/2$. Up to an infinite quantity, the scattering amplitude (9) is expandable in partial waves, and the corresponding amplitudes satisfy the optical theorem. If the sign of α changes (repulsive or attractive potentials), the scattering amplitude gets complex conjugate.

For identical particles we note that an interference term appears in $|f(\theta) \pm f(\pi-\theta)|^2$, of the form $\cos[\frac{1}{kD}\ln\tan^2\theta/2]$.

If an additional potential is present, as the nuclear potential for instance, then it contributes its own scattering amplitude. For s-wave at resonance, it reads $f_0 = a/(1-iak)$, where a is the (large) scattering length (and $k \sim 0$). This scattering amplitude can also be written as $f_0 = ae^{i\delta}$ for $ak \sim \delta$, such that an interference term appears in the scattering cross-section, of the form $\cos\{\frac{2}{kD}\ln\sin\theta/2 + \delta\}$. This interference term may serve to correct the additional scattering length a (through δ) by terms of the form $(1/D)\ln D$, for right-angle scattering where $D\sin^2\theta/2 \sim const$.

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