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Inelastic collisions

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Abstract

The theory of the inelastic collisions with applications in nuclear physics is briefly reviewed.

The scattering (S) matrix. The scattering amplitude in the asymptotic scattering wave

$$e^{i\mathbf{kr}} + f \cdot \frac{e^{ikr}}{r} \tag{1}$$

can be viewed as a matrix $f(\mathbf{n},\mathbf{n}')$ for unit vectors $\mathbf{n} = \mathbf{k}/k$ and $\mathbf{n}' = \mathbf{r}/r$, defined as $f \rightarrow (1/4\pi)\int d\Omega \cdot f(\mathbf{n},\mathbf{n}')\cdot$ A similar integration of the plane wave over angles takes (1) into

$$\frac{e^{-ikr}}{r} - S \cdot \frac{e^{ikr}}{r} \ , \tag{2}$$

where

$$S = 1 + 2ikf \ . \tag{3}$$

This S in equation (3) is the scattering, or the S-, matrix.[1]

For elastic scattering $SS^* = 1$ in order to conserve the number of particles, so that $Imf(\mathbf{n},\mathbf{n}) = Imf(0) = k\sigma/4\pi$, where $\sigma = \int d\Omega \cdot |f|^2$ is the cross-section. This is the optical theorem. For central potentials the partial-wave amplitudes are therefore of the form $f = 1/(g - ik)$, where g is a real function of k, as, for instance, $f_0 = 1/(-a^{-1} - ik)$ for the s-wave, where a is the scattering length. Making use of (3) we get $S = (g + ik)/(g - ik) = \exp(2i\delta)$, where δ is the phase shift, with $\tan\delta = k/g$.

Inelastic scattering. Inelastic scattering changes the internal states of the colliding particles, even destroying or creating some. Beside the elastic scattering, there are more reaction channels of inelastic scattering. S has not modulus unity anymore, and function g is now complex. The partial-wave cross-section for elastic channel is $\sigma_e = (\pi/k^2)\,|1 - S|^2$. The partial-wave cross-section for reaction channels are obtained from (2) as $\sigma_r = (\pi/k^2)(1 - |S|^2)$ (because the outgoing wave has the coefficient S), such that the total cross-section is given by $\sigma_t = \sigma_e + \sigma_r = (2\pi/k^2)(1 - ReS)$. For $S = 1$ there is no scattering, for $S = 0$ there is only total absorption (no outgoing wave in (2)), and $|S| < 1$. One can see by (3) that the optical theorem reads now $Imf(0) = k\sigma_t/4\pi$, and the partial-wave amplitudes read $f = 1/(g - ik)$, where g is now a complex function of k^2. Indeed, making $k \rightarrow -k$ in (2) one can see that $S \rightarrow 1/S$, so that, eliminating S in (3) and the corresponding equation for $-k$, it is easy to get function g as $g(k^2)$.

[1] W. Heisenberg, Z. Phys. **120** 513, 673 (1943).

Attenuation. High-energy beams of particles are attenuated through inhomogeneous media by forward scattering of amplitude, say, f. For particle wavelengths much shorter than the average inter-particle distance in the medium the scattering proceeds by individual scatterers. For f much shorter than the inter-particle distance in the medium an effective potential V may give f through Born formula $f = (m/2\pi\hbar^2)\int dr \cdot V$, where m is the mass of beam particles. A weak average potential $\bar{V} = (N/V)\int dr \cdot V = (2\pi\hbar^2 f/m)(N/V)$ acts in the medium, where N/V is the particle density of the medium.

Such a potential is felt by highly excited alkali atoms in a gas, and the excitation level of the valency electron is shifted by $(2\pi\hbar^2 f/m)(N/V) = -(2\pi\hbar^2 a/m)(N/V)$, where $f = -a$ is the scattering length, since the valency electron is slow.

The propagation wavevector is given by $k = \sqrt{2m(E-\bar{V})/\hbar^2} \simeq k_0(1-\bar{V}/2E)$, where $k_0 = 2mE/\hbar^2$. A refractive index n can be defined through $k = nk_0$, and $n = 1-\bar{V}/2E = 1-(\pi\hbar^2 f/mE)(N/V)$. Its imaginary part is given by $Imn = (\pi\hbar^2 Imf/mE)(N/V) = (2\pi Imf/k_0^2)(N/V) = = (\sigma_t/2k_0)(N/V)$, by the optical theorem. The beam wave goes like $\exp(ikz) \sim \exp[-\sigma_t(N/2V)z]$.

High-energy beams of fast neutrons with energy $\sim 100MeV$ can be atenuated in this way on their passing through the atomic nuclei. Fast-particles scattering proceeds classically, the total cross-section is $4\pi\int d\rho \cdot \rho = 2\pi a^2$, where ρ is the impact parameter and a is the scattering sphere radius. Fast particles have one chance in two to be absorbed or elastically scattered, so that $\sigma_e = \sigma_r = \pi a^2$. This is the Fraunhofer diffraction by a sphere.

Scattering of slow particles. Slow particles scatter in s-wave providing $kb \ll 1$, where b is the range of the potential. (For Coulomb potential this b is replaced by Colomb length $D = \hbar^2/me^2$, where e is the (common) charge of the colliding particles).[2] The scattering amplitude is $f = 1/(-\alpha^{-1} - ik) = -\alpha/(1+i\alpha k)$, where $\alpha = \alpha' + i\alpha''$ is the scattering length, now complex. The phase factor reads $S = (1-i\alpha k)/(1+i\alpha k)$ and $|S|^2 = (1+2\alpha'' k+|\alpha|^2 k^2)/(1-2\alpha'' k+|\alpha|^2 k^2)$, such that $\alpha'' < 0$ since $|S| < 1$. The elastic cross-section is $\sigma_e = 4\pi|\alpha|^2$, while the inelastic (reaction) cross-section is given by $\sigma_r = 4\pi|\alpha''|/k$ for $|\alpha| k \ll 1$. This is the $1/v$-law of inelastic scattering, which tells that the reaction cross-section increases for slow particles, with vanishing velocity v. (Similarly, for non-vanishing angular momenta l, $f \sim -\alpha \sim k^{2l}$, and $\sigma_e \sim k^{4l}$, $\sigma_r \sim k^{2l-1}$).

In general, the probability to have a reaction is proportional to the ratio of the square modulus of the wavefunction in the reaction range $|\psi(r \sim 0)|^2$ and the current density at infinity $\sim v|\psi|^2$, which explains the $1/v$-law for slow particles, as, for instance, slow neutrons.

For repulsive Coulomb potential $f \sim e^{-1/kD}$ since $k \rightarrow ik$ in the reaction zone, [3] and the ratio of the two square moduli of the wave functions is $\sim e^{-2/kD}/(1-e^{-2/kD}) \sim 1/(e^{2/kD}-1)$. The reaction cross-section is therefore $\sigma_r \sim 1/k^2(e^{2/kD}-1)$. For high energy $kD \gg 1$ we recover the $1/v$-law $\sigma_r \sim 1/k$, as expected, since the Coulomb scattering plays no role in this limit. For slow particles $kD \ll 1$ the reaction rate is exponentially small, $\sigma_r \sim e^{-2/kD}/k^2$.

For attractive Coulomb potential $f \sim e^{1/kD}$ in the reaction zone, and $\sigma_r \sim 1/k^2\left|e^{-2/kD}-1\right|$. For $kD \gg 1$ we recover the $1/v$-law, but for $kD \ll 1$ we get the limiting law $\sigma_r \sim 1/k^2$. The exponential factor $e^{2/kD}$ by which the repulsive potential differs from the attractive one for slow particles is their probability of passage through the Coulomb potential barrier.

Reaction channels. Let i denote the input (elastic) channel, and f the output reaction (inelastic) channels. The scattering amplitudes are given by the asymptotic waves

$$\psi_i = e^{ik_i z} + f_{ii} \cdot \frac{e^{ik_i r}}{r} \quad , \tag{4}$$

[2] M. Apostol, J.Theor. Phys. **128**, **131** (2006).
[3] M. Apostol, J. Theor. Phys. **131** (2006).

and, respectively,

$$\psi_f = f_{fi}\sqrt{m_f/m_i}\cdot\frac{e^{ik_f r}}{r} \ , \tag{5}$$

where the relative masses $m_{i,f}$ are introduced for normalization reasons. Indeed, the cross-sections read $d\sigma_{ii} = |f_{ii}|^2 \, d\Omega_i$ and $d\sigma_{fi} = (p_f/p_i)\, |f_{fi}|^2 \, d\Omega_f$, as defined by the ratio of the flows.

In order to introduce the generalized S-matrix, an angle integration must be performed in (4), as in equations (1) to (3). We get

$$\psi_i = \frac{e^{-ik_i r}}{\sqrt{v_i}r} - (1+2ik_i f_{ii})\cdot\frac{e^{ik_i r}}{\sqrt{v_i}r} \ , \tag{6}$$

and

$$\psi_f = 2ik_i f_{fi}\sqrt{m_f/m_i}\cdot\frac{e^{ik_i r}}{\sqrt{v_i}r} \ , \tag{7}$$

where a further $\sqrt{v_i}$ factor is introduced for having unit flows. It follows immediately the generalized S -matrix

$$S_{fi} = \delta_{if} + 2i\sqrt{k_i k_f} f_{fi} \ . \tag{8}$$

The unitarity condition reads

$$f_{fi} - f_{if}^* = 2i\sum k_n f_{fn} f_{nf}^* \ , \tag{9}$$

where summation is over all scattering states.

For partial-wave elastic scattering the relevant factors in cross-sections are $|1-S_{ii}|^2$, while for reaction cross-sections they are $|S_{fi}|^2$. Since the matrix is unitary, $\sum |S_{fi}|^2 = 1 - |S_{ii}|^2$, so that we recover the reaction cross-section $\sigma_r = \sum' \sigma_{fi} = (\pi/k_i^2)(1-|S_{ii}|^2)$.

The amplitudes are symmetric under time reversal t, whereby $i \longleftrightarrow f$ and momenta and internal angular momenta change sign. This symmetry implies

$$p_i^2 d\sigma_{fi} d\Omega_f = p_f^2 d\sigma_{if}^t d\Omega_i \ , \tag{10}$$

which is the reciprocity theorem or the principle of detailed balancing. An average over spin states of the cross-sections are sometime useful.

Partial widths. The well-known Breit-Wigner formula for resonance scattering[4] has been rederived[5] with the result

$$f = -\frac{\Gamma/2k}{E - E_0 + i\Gamma/2} \ . \tag{11}$$

They represent a special form of scattering amplitudes, corresponding to resonance on quasi-discrete levels of a complex particle, like, for instance the compound nucleus.[6] They are both elastic (diagonal in the input channel) and inelastic (non-diagonal) amplitudes.

We define the generalized (multi-channel) amplitudes as

$$f_{ab} = \frac{1}{2ik_a}(e^{2i\delta_a}-1)\delta_{ab} - \frac{1}{2\sqrt{k_a k_b}} e^{i(\delta_a+\delta_b)}\frac{\Gamma M_{ab}}{E-E_0+i\Gamma/2} \ , \tag{12}$$

for a resonance scattering on level E_0, of width Γ, with multiple channels, including the input non-resonant channel, for a definite angular momentum l, where M_{ab} remain to be determined.

[4] G. Breit and E. P. Wigner, Phys. Rev. **49** 519, 642 (1936).
[5] M. Apostol, J. Theor. Phys. **128** (2006).
[6] N. Bohr, Nature **137** 344 (1936).

Spins are left aside. The unitary condition (9) leads to the conclusion that M is a symmetric matrix with unity eigenvalues, so it can be expressed as $M_{ab} = \sqrt{\Gamma_a \Gamma_b}/\Gamma$, where

$$\sum \Gamma_a = \Gamma \tag{13}$$

are partial widths (for a non-degenerate quasi-discrete level).

Let Γ_e be the partial width of the input elastic channel and $\Gamma_{r1,2,...}$ the partial widths of the reaction channels. The total elastic amplitude given by (12) reads

$$f_e = f^{(0)} - \frac{2l+1}{2k} e^{2i\delta} \frac{\Gamma_e}{E - E_0 + i\Gamma/2} P_l(\cos\theta) \ , \tag{14}$$

where $f^{(0)}$ is the potential amplitude and P_l are the Legendre polynomials. The reaction processes are purely of resonance type, so that the corresponding cross-section reads

$$d\sigma_{ra} = \frac{(2l+1)^2}{4k^2} \frac{\Gamma_e \Gamma_{ra}}{(E-E_0)^2 + \Gamma^2/4} P_l^2(\cos\theta) d\Omega \ . \tag{15}$$

Similarly, the integral cross-section is given by

$$\sigma_{ra} = (2l+1) \frac{\pi}{k^2} \frac{\Gamma_e \Gamma_{ra}}{(E-E_0)^2 + \Gamma^2/4} \ , \tag{16}$$

and the total reaction cross-section reads

$$\sigma_r = (2l+1) \frac{\pi}{k^2} \frac{\Gamma_e \Gamma_r}{(E-E_0)^2 + \Gamma^2/4} \ , \tag{17}$$

where $\Gamma_r = \Gamma - \Gamma_e$. It can also be integrated over energy to give

$$\int dE \cdot \sigma_r = (2l+1) \frac{2\pi^2}{k^2} \cdot \frac{\Gamma_e \Gamma_r}{\Gamma} \ . \tag{18}$$

Slow neutrons. Slow neutrons scatter from nucleus in s-wave, so the elastic amplitude (14) reads

$$f_e = -a - \frac{\Gamma_e}{2k(E - E_0 + i\Gamma/2)} \ , \tag{19}$$

where $-a$ is the scattering length. The total elastic scattering cross-section is given by

$$\sigma_e = 4\pi a^2 + \frac{\pi}{k^2} \cdot \frac{\Gamma_e^2 + 4ak\Gamma_e(E-E_0)}{(E-E_0)^2 + \Gamma^2/4} \ . \tag{20}$$

The first term is the potential scattering, while the second one represents the resonance scattering. There is an interference between these two terms, in the resonance region. Close to resonance the potential scattering may be neglected, and we get

$$\sigma_e = \frac{\pi}{k^2} \cdot \frac{\Gamma_e^2}{(E-E_0)^2 + \Gamma^2/4} \ . \tag{21}$$

The inelastic cross-section is given by

$$\sigma_r = \frac{\pi}{k^2} \cdot \frac{\Gamma_e \Gamma_r}{(E-E_0)^2 + \Gamma^2/4} \ , \tag{22}$$

and the total cross-section reads

$$\sigma_t = \sigma_e + \sigma_r = \frac{\pi}{k^2} \cdot \frac{\Gamma_e \Gamma}{(E - E_0)^2 + \Gamma^2/4} \ , \tag{23}$$

since $\Gamma = \Gamma_e + \Gamma_r$. These cross-sections can also be represented as

$$\sigma_e = \sigma_t \Gamma_e / \Gamma \ , \ \ \sigma_r = \sigma_t \Gamma_r / \Gamma \ , \tag{24}$$

which shows that the partial widths are probabilities, in agreement with their definition.

These formulae are valid for $|E - E_0| \ll D$, where D is the separation in the quasi-discrete resonance levels. Similarly, they do not apply for a resonance with vanishing energy. There, a special form of the Wigner-Breit formula must be used, which agrees with the general resonance elastic scattering.[7]

Spins modify the formulae above. Each quasi-discrete level has an angular momentum $j = i \pm 1/2$, where i is the spin of the nucleus and $1/2$ is the spin of the neutron. For random spins, the weight $g = (2j + 1)/2(2i + 1)$ must be included in the cross-sections, including the potetial scattering (where both j-values contribute).

Optical model. For many resonances it is preferable to employ an S averaged over energies. This way, we are left with elastic and inelastic collisions, while various reaction channels are lost. The definitions of the cross-sections are preserved, *i.e.* $\sigma_e = (\pi/k^2) \left|1 - \bar{S}\right|^2$ for the elastic cross-section, $\sigma_r = (\pi/k^2)(1 - \left|\bar{S}\right|^2)$ for the reaction (inelastic) cross-section, and $\sigma_t = \sigma_e + \sigma_r = (2\pi/k^2)(1 - Re\bar{S})$ for the total cross-section. Details of the elastic cross-section are lost, according to the average procedure, which justify the term of "coherent" scattering. In particular, the part of elastic scattering through the formation of the compound nucleus is now transferred to the inelastic cross-section.

For low energies, where the quasi-discrete levels are well-resolved, the S-matrix reads

$$S = e^{2i\delta}(1 - \frac{i\Gamma_e}{E - E_0 + i\Gamma/2}) \ , \tag{25}$$

by $S = 1 + 2ikf$. Averaging over energies gives

$$\bar{S} = e^{2i\delta}(1 - \pi\bar{\Gamma}_e/D) \ , \tag{26}$$

so that the the total cross-section is given by

$$\sigma_t = \frac{\pi}{k^2} \cdot \frac{2\pi\bar{\Gamma}_e}{D} \ ; \tag{27}$$

It coincides in fact with the averaged σ_t given by (23). At high energies the averaged quantities get close to the exact ones. An effective, complex maen-field potential may serve in the optical model to represent the average S-matrix, according to the formulae given above.

Reaction threshold. The final reaction products may require a certain threshold energy E_t to be produced. According to the principle of the detailed balancing, for energies E slightly above this threshold the cross-section is proportional to v'^2 multiplied by the time-reversed cross-section, the latter going like $1/v'$, where $v' \sim \sqrt{E - E_t}$ is the relative velocity of the reaction products.

[7] M. Apostol, J. Theor.Phys. **128** (2006).

Therefore, for short-range potentials, the threshold reaction cross-section is $\sigma_r \sim 1/v' \sim \sqrt{E-E_t}$. Comparing it with $\sigma_r \sim (\pi/k^2)(1-|S|^2)$ for s-wave scattering, one obtaines

$$|S| = 1 - A(k_t^2/2\pi)\sqrt{E-E_t} \ , \tag{28}$$

where $E_t = \hbar^2 k_t^2/2m$ and A is a constant. Let us suppose that below the threshold there exists only elastic scattering, as, for instance, in production of slow neutrons. Then, $|S| = 1$ for $E < E_t$, and

$$S = e^{2i\delta}[1 - A(k_t^2/2\pi)\sqrt{E-E_t}] \tag{29}$$

for all energies close to E_t, to the same accuracy. The scattering amplitude is therefore given by

$$f = f_t - \frac{k_t}{4\pi i} A\sqrt{E-E_t} e^{2i\delta} \ , \tag{30}$$

where f_t is the scattering amplitude at the threshold. The differential cross-section is then given by

$$d\sigma/d\Omega = |f_t|^2 + \frac{k_t}{2\pi} A\sqrt{E-E_t} Im(f_t e^{-2i\delta}) \ , \ \ E > E_t \ , \tag{31}$$

and

$$d\sigma/d\Omega = |f_t|^2 - \frac{k_t}{2\pi} A\sqrt{E-E_t} Re(f_t e^{-2i\delta}) \ , \ \ E < E_t \ . \tag{32}$$

Depending on the real and imaginary parts in (32) the cross-section exhibits characteristic cups at the threshold. If spin is included, the resulting s-wave may arise from various j-values of the original reacting particles, but the general cusp-like behaviour of the cross-section is preserved. This holds also for vaious other reactions below the threshold.

All of this above holds for short-range potentials. For Coulomb potentials, as for instance, in production of protons (repulsive Coulomb interaction) the cross-section tends exponentially to zero above the threshold ($\sigma_r \sim e^{-2/kD}/k^2$, see above), and there is no singularity.

For attractive Coulomb interaction, the cross-section has a sudden, constant jump above the treshold ($\sigma_r \sim 1/k^2$, see above). Below the threshold, there must be resonances (similar to Wigner-Breit ones), due to quasi-discrete levels correponding to bound states in attractive Coulomb potential. Making use of the principle of detailed balancing and of the cross-section for attractive Coulomb interaction ($\sigma_r \sim 1/k^2(e^{2/kD}-1)$, see above), we get $\sigma \sim 1/\sin(1/\kappa D)$, by changing $k \rightarrow i\kappa \sim i\sqrt{E_t - E}$. This cross-section exhibits oscillations below threshold, the faster the closer the threshold.[8]

Interaction in the final state of reaction. If two slow nucleons are created in the final state, the nuclear forces may affect their cross-section (probability of their occurrence), either by their possible discrete or virtual level. The probability is given by $dw = |\psi|^2 d\mathbf{p}$ in the reaction region, where $\mathbf{p}$ is their relative momentum, much smaller than the momentum $\mathbf{p}_0$ of the center of mass. The corresponding energies are E and, respectively, E_0. The wavefunction is given by $[1/(\kappa + ik)]e^{ikr}/r$, where the well-known resonance scattering amplitude for small energies is employed. It leads to $dw \sim d\mathbf{p}/(|\varepsilon| + E)$, where $\varepsilon < 0$ is a discrete level and $\varepsilon > 0$ is a virtual level. All the same, $dw \sim \sqrt{E}dE/(|\varepsilon| + E)$. We can see that the cross-section has a maximum for $E \sim |\varepsilon|$.

[8]Such oscillations are related to the ionization of a neutral atom by an electron with an energy E sligtly above the ionization threshold I. This process has a cross-section $\sim (E-I)^\alpha$, where $\alpha \simeq 1$ (G. H. Wannier, Phys. Rev. **90** 817 (1953)).

In the laboratory frame the momenta are $\mathbf{p}_0 = \mathbf{p}_1 + \mathbf{p}_2$ and $\mathbf{p} = (\mathbf{p}_2 - \mathbf{p}_1)/2$. We have $\mathbf{p}_0 \times \mathbf{p} = \mathbf{p}_1 \times \mathbf{p}_2$, and $p_0 p_\perp = p_1 p_2 \sin\theta = p_0^2\theta/4$, since $p_0 \gg p$; $\theta = 4p_\perp/p_0 \ll 1$ is the angle between the two momenta $\mathbf{p}_{1,2}$ and $p_\perp$ is the transverse component of $\mathbf{p}$. The probability distribution becomes

$$dw \sim \frac{p_\perp dp_\perp dp_\parallel}{(p_\perp^2 + p_\parallel^2)/m + |\varepsilon|} \tag{33}$$

or, integrating over $p_\parallel$, we get

$$dw \sim \frac{\theta d\theta}{\sqrt{\theta^2 + 4\,|\varepsilon|\,/E_0}} \ ; \tag{34}$$

it has a maximum for nearly-forward scattering $\theta \sim \sqrt{|\varepsilon|\,/E_0}$. It may give $|\varepsilon|$.