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Twins paradox

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Abstract

Asking a question of my firend M Ganciu. This note stems from his question.

Space $d\mathbf{r}$ and time dt must relate to each other. There must exist therefore a universal velocity c, and the relation must imply $d\mathbf{r}^2$ and c^2dt^2 in order to account for the two signs of the velocity. For light $ds^2 = c^2dt^2 - d\mathbf{r}^2 = 0$, so it remains zero in any other frame of linear transforms of coordinates. It follows that $ds^2 = c^2dt^2 - d\mathbf{r}^2$, where c is the velocity of light, is invariant for all motions, and these motions proceed with a velocity $v < c.$. These are the basic ideas of relativity. The velocity of light c is universal, and ds^2 is invariant under linear transforms of coordinates. For constant velocity we get the principle of relativity for inertial frames. For non-uniform velocity, as well as for curved spaces, the length ds is given in general by $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, where $g_{\mu\nu}$ is the metric tensor and $x^\mu = ct, \mathbf{r}$. In general, the metric tensor is local, as for curved spaces.

The Lorentz tranforms

$$dx = \frac{dx' + vdt'}{\sqrt{1 - v^2/c^2}} \ , \ dt = \frac{dt' + vdx'/c^2}{\sqrt{1 - v^2/c^2}} \tag{1}$$

follow immediately from the invariance of length ds, for a motion along x with velocity v.

They can also de deduced in another way. Indeed, if x and t are going to depend each on x' and t', the only admissible linear combinations are $x = f(x' + vt')$ and $t = g(t' + vx'/c^2)$, where f and g are functions of v^2. For light, these combinations become $x = f(x' + ct')$, $t = g(t' + x'/c)$, which implies $f = g$ since $x = ct$. We then apply these two transforms successively for v and $-v$. We get $x = f[f(x'' + vt'') - vf(t'' + vx''/c^2)] = f^2(1 - v^2/c^2)x'' = x''$, whence $f = (1 - v^2/c^2)^{1/2}$.

Let $t' = \tau$ be the proper time of a moving object, say observer A, placed at the origin, $dx' = 0$. We get

$$dt = \frac{d\tau}{\sqrt{1 - v^2/c^2}} \ , \tag{2}$$

and the proper time is shorter for moving bodies. Motion dilates the time. Similarly, if observer A measures the time for the frame at rest, say observer B, the latter will move with velocity $-v$ with respect to the former, so observer A will see a dilated time for observer B, as given by the same equation (2).

It is of utmost importance to stress upon the fact that the measured object in the former case is observer A, while the measured object in the latter case is observer B. Because, the difference in the two clocks is the same, but the measurements refer to two distinct things. There is a fundamental difference between the time the observer measures for himself and the time the same observer measures for another, moving, object. This is the main content of the principle of relativity.

Suppose that observer A moves away from observer B, and then it comes back, like in a galactic journey. The two observers, one after such a journey the other at rest, compare their clocks. The clocks tell different times, as given by

$$t = \int_0^\tau \frac{d\tau}{\sqrt{1 - v^2/c^2}} \ . \tag{3}$$

Let $v = v_0 \sin \omega\tau$, as describing such a journey, and $v_0/c \ll 1$. We get easily from (3)

$$t = \tau(1 + v_0^2/4c^2) \ , \tag{4}$$

where $\omega\tau = 1$. The clock of the traveller A have lost, and it will tell a shorter time. Similarly, if the observed object is observer B, its clock will then be the one which tells a shorter time.

This is the twins paradox. After journey, observer A will be younger than B, according to the measurement observer B makes upon A. At the same time, B will be younger than A, if B is observed by A. The time depends on the observer. The proper time is the same for both observers.