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**On certain dimensionality effects in the ideal Bose and Fermi gases**

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**Abstract**

A recently established equivalence between the ideal Bose and Fermi gases (M. H. Lee, Phys. Rev. **E55** 1518 (1997)) is shown to be a peculiarity of the boson-like correlations in two dimensions; such an equivalence does not hold in one or three dimensions.

Recently,[1] Lee established a remarkable equivalence between the ideal Bose and Fermi gases in two dimensions. The equivalence is based on a certain invariance of the polylogs[2] (see also Ref.3) under Euler′s transform of the fugacities of the two gases, an invariance found many years ago by Landen.[4] The result might have been expected, since a non-relativistic two-dimensional gas is equivalent with a relativistic gas in one dimension. We show in this paper that such an equivalence does not exist in one or three dimensions. The reason for this remarkable particularity resides in the combined effect of the boson-like correlations (responsible for the Bose-Einstein condensation) and dimensionality.

We begin with a brief review of Lee′s result. The basic object is the number of "thermal states"

$$\nu = \frac{N\lambda^2}{gA} \quad , \tag{1}$$

where $N$ is the number of particles, $A$ is the area occupied by the gas, $g$ is a kinematical factor of degeneracy, and

$$\lambda = \left(\frac{2\pi\hbar^2}{mT}\right)^{1/2} \tag{2}$$

is the thermal wavelength; in (2) $\hbar$ is Planck′s constant, $m$ is the mass of a particle, and $T$ $(= 1/\beta)$ is the temperature. Introducing the inter-particle spacing $a = (A/N)^{1/2}$ and the characteristic energy $\varepsilon_0 = 2\pi\hbar^2/gma^2$ we get $\nu = \varepsilon_0/T$, which justifies the designation number of thermal states; for a Fermi gas $\varepsilon_0 = 2\varepsilon_F$, where $\varepsilon_F$ is the Fermi energy. The number of thermal states is given by $\nu_b = -\ln(1 - z_b)$ for bosons, and $\nu_f = \ln(1 + z_f)$ for fermions, where $z_{b,,f}$ are the fugacities. For $\nu_b = \nu_f = \nu$ we get $1 + z_f = (1 - z_b)^{-1}$, which is precisely Euler′s transform between $z_b$ and $-z_f$. We note that $z_b = 1 - \exp(-\nu_b)$ and $z_f = \exp(\nu_f) - 1$. The energies of the two gases are given by

$$\begin{aligned} \beta\nu_b E_b/N_b &= Li_2(z_b) \quad , \\ \beta\nu_f E_f/N_f &= -Li_2(-z_f) \quad , \end{aligned} \tag{3}$$

where $Li_2(z)$ is the dilog of $z$. A useful integral representation of the polylogs is [2]

$$Li_{n+1}(z) = \frac{1}{\Gamma(n+1)} \int_0^z du \cdot \left(\ln \frac{z}{u}\right)^n \frac{1}{1-u} \tag{4}$$

for $Rez < 1$. Under Euler′s transform between $z_b$ and $-z_f$ given above the dilog becomes

$$L_2(z_b) = -Li_2(-z_f) - \frac{1}{2} Li_1^2(-z_f) \quad , \tag{5}$$

which is precisely Landen′s relation.[1],[4] Using (5) we obtain straightforwardly

$$E_b/N_b = E_f/N_f - \frac{1}{2}\varepsilon_F \quad , \tag{6}$$

which is Lee′s main result. In addition, it follows from (6) that the specific heats of the two gases are equal, a result previously established.[5] Since $\Omega = -E$ in two dimensions,[6] where the thermodynamic potential $\Omega = -(1/\beta)\ln Q = -pA$, $Q$ being the grand-partition function and $p$ being the pressure, we have also

$$p_b/n_b = p_f/n_f - \frac{1}{2}\varepsilon_F \quad , \tag{7}$$

and the equality of the entropies, $S_b/N_b = S_f/N_f$. These relations establish a perfect equivalence between the ideal Bose and Fermi gases in two dimensions.

We pass now to show that such an equivalence does not exist in one or three dimensions. Naturally, we shall be interested in high values of the number of thermal states $\nu$, $\nu \gg 1$, *i.e.* in temperatures much lower than the degeneracy temperature. In three dimensions we have $\nu = (\varepsilon_0/T)^{3/2}$ and $\varepsilon_0 = (4/9)^{1/3}\varepsilon_F$ for the Fermi gas. For an ideal Fermi gas the number of thermal states is given by

$$\nu_f = -\Gamma\left(3/2\right) Li_{3/2}(-z_f) \quad , \tag{8}$$

and making use of the well-known integrals with the Fermi-Dirac distribution we get[7]

$$\nu_f = \frac{2}{3}\left(\ln z_f\right)^{3/2}\left[1 + \frac{\pi^2}{8}\frac{1}{\left(\ln z_f\right)^2} + ...\right] \quad ; \tag{9}$$

we see that $z_f \gg 1$ for $\nu_f \gg 1$. Similarly, for an ideal Bose gas we have

$$\nu_b = \Gamma\left(3/2\right) Li_{3/2}(z_b) \quad . \tag{10}$$

However, in contrast to the two-dimensional case, an interesting phenomenon occurs in three dimensions as a result of the boson-like correlations, a phenomenon which is in fact the Bose-Einstein condensation.[8] Indeed, a simple change of variable in (4) leads to

$$Li_{3/2}(z_b) = \sum_{n=1}^{\infty} \frac{z_b^n}{n^{3/2}} \quad . \tag{11}$$

For $0 < z_b < 1$ this series is bounded by $Li_{3/2}(1) = \zeta(3/2)$, where $\zeta$ is Riemann′s zeta function. Therefore, $\nu_b$ is bounded by $\Gamma(3/2)\zeta(3/2)$, which means that the bosons condense on the zero-energy level. Consequently, $\nu_b$ can not be equal to $\nu_f$, and the two gases ar not equivalent.

A similar situation appears in one dimension, though not for the number of thermal states, but for the energy.The number of thermal states in one dimension is $\nu = (\varepsilon_0/T)^{1/2}$, where $\varepsilon_0 = 4\varepsilon_F$ for the Fermi gas. For an ideal Fermi gas in one dimension we have

$$\nu_f = -\Gamma\left(1/2\right) Li_{1/2}(-z_f) \quad , \tag{12}$$

and

$$\nu_f = 2\left(\ln z_f\right)^{1/2}\left[1 - \frac{\pi^2}{24}\frac{1}{\left(\ln z_f\right)^2} + ...\right] \tag{13}$$

in the asymptotic regime $\nu_f, z_f \gg 1$. Similarly the energy is given by

$$\beta\nu_f E_f/N_f = -\Gamma\left(3/2\right) Li_{3/2}(-z_f) = \frac{2}{3}\left(\ln z_f\right)^{3/2}\left[1 + \frac{\pi^2}{8}\frac{1}{\left(\ln z_f\right)^2} + ...\right] \quad , \tag{14}$$

whence

$$E_f/N_f = \frac{1}{3}\varepsilon_F\left(1 + \frac{\pi^2}{4}\frac{T^2}{\varepsilon_F^2} + ...\right) \quad . \tag{15}$$

From (15) we obtain the well-known specific heat of an ideal Fermi gas

$$c_f = \frac{\pi^2}{6}\frac{T}{\varepsilon_F} \quad . \tag{16}$$

For an ideal Bose gas in one dimension we have

$$\nu_b = \Gamma\left(1/2\right) Li_{1/2}(z_b) = \sqrt{\pi}\sum_{n=1}^{\infty}\frac{z_b^n}{\sqrt{n}} \quad , \tag{17}$$

and the series given by (17) diverges for $z_b \rightarrow 1$. Therefore, we could have a relationship between $z_b$ and $z_f$ for $\nu_b = \nu_f = \nu \gg 1$. However, the energy of the Bose gas is given by

$$\beta\nu_b E_b/N_b = \Gamma\left(3/2\right) Li_{3/2}(z_b) = \Gamma(3/2)\sum_{n=1}^{\infty}\frac{z_b^n}{n^{3/2}} \quad , \tag{18}$$

and we see again that the energy per particle is now bounded, in contrast to the Fermi case. Moreover, for $z_b \rightarrow 1$ we get

$$E_b/N_b \cong \frac{1}{4}\sqrt{\frac{\pi}{\varepsilon_F}}\zeta(3/2)\cdot T^{3/2} \quad , \tag{19}$$

and comparing it with (15) we see that there can be no equivalence in one dimension, of the type established by Lee in two dimensions.

The arguments presented above can be summarized as follows. The number of thermal states for fermions in dimension $d$ is given by

$$\nu_f^{(d)} \sim \int_0^\infty dx \cdot x^{d/2-1}\frac{z_f}{e^x + z_f} \quad , \tag{20}$$

and in the asymptotic regime $\nu_f \gg 1,\ z_f \rightarrow \infty$, we obtain

$$\nu_f^{(d)} \sim \left(\ln z_f\right)^{d/2} \quad . \tag{21}$$

For an ideal gas of bosons $\nu_b \gg 1$ corresponds to $z_b \to 1$, and the boson-like correlations expressed by the singularity of the Bose-Einstein distribution at vanishing energy determine distinct asymptotic behaviours of $\nu_b$ with dimension $d$. In three dimensions $\nu_b^{(3)}$ is finite for $z_b \to 1$, indicating the Bose-Einstein condensation; thus, there can not be any equivalence between bosons and fermions in this case. In one dimension $\nu_b^{(1)}$ diverges for $z_b \to 1$,like $\nu_b^{(1)} \sim (1-z_b)^{-1/2}$, and a relationship with $\nu_f^{(1)}$ given by (21) might be possible; however, the energy per particle in this case goes like $E_b/N_b \sim T^{3/2}$ for bosons, while $E_f/N_f \sim const + T^2$ for fermions, according to (15) and (19), and we see again that there could not be any equivalence. In two dimensions $\nu_b^{(2)} = -\ln(1-z_b)$ and comparing it with (21) we can see that such an equivalence might be possible via Euler′s transform between the two fugacities; in adition $E_b/N_b \sim T^2$ and $E_f/N_f \sim const + T^2$ in this case, which makes this equivalence even more likely. However, its precise demonstration, as given by Lee[1], remains a remarkable property of the ideal Bose and Fermi gases in two dimensions.

# References

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[6] Making use of (3) and expressing the potential $\Omega$ directly with the dilog we obtain also the addition formula $Li_2(1-z) + Li_2(z) = Li_2(1) - Li_1(z) \cdot Li_1(1-z)$.

[7] By the same method we can obtain the asymptotic formula

$$Li_{n+1}(-z) = -\frac{1}{(n+1)\,\Gamma(n+1)}\,(\ln z)^{n+1}\left[1 + \frac{\pi^2}{6}n(n+1)\frac{1}{(\ln z)^2} + ...\right]$$

for $z \gg 1$

[8] As it is well-known there is no superfluid transition in two dimensions; see, for example, M. F. M. Osborne, Phys. Rev. **76** 396 (1949).