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Elementary comments on unitary transformations

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Let ψ, φ... denote quantum states. U is a unitary transformation if it preserves the scalar products, *i.e.* $(U\varphi, U\psi) = (\varphi.\psi)$. The probabilities are then preserved, so unitary transformations describe symmetries. Probabilities are also preserved by $(U\varphi, U\psi) = (\varphi, \psi)^*$, and U is then called an anti-unitary transformation, which is also associated with a symmetry. This is a Wigner theorem.

Obviously a unitary transformation is such that $UU^+ = U^+U = 1$, so it is representable by $U = e^{iS}$, where S is hermitian, $S^+ = S$, and, therefore, an observable in principle.

A unitary transformation preserves the commutation relations, *i.e.* the laws of motion, so it is also a canonical transformation.

Beside symmetries, another main application of the unitary transformations is their removing of interaction. Indeed, let the hamiltonian be $H = H_0 + \varepsilon V$, with usual notations, and let $\psi = e^{i\varepsilon S}\psi'$. Then $H'\psi' = e^{-i\varepsilon S} H e^{i\varepsilon S}\psi' = H_0\psi' + O(\varepsilon^2)$, providing $i\,[H_0, S] = V$. The effect of the interaction is now transferred upon the state $\psi = e^{i\varepsilon S}\psi'$. The interaction may depend on time, so S depends on time, and Schrodinger's equation requires

$$\partial S_{nk}/\partial t + i\omega_{nk} S_{nk} + V_{nk}/\hbar = 0 \ , \quad (1)$$

where $\omega_{nk} = (E_n - E_k)/\hbar$. It follows

$$S_{nk} = (1/\hbar) \int^t dt' \cdot V_{nk}(t') e^{i\omega_{nk}t'} \cdot e^{i\omega_{nk}t} \ , \quad (2)$$

which is the amplitude of transition from state ψ_k' into state ψ_n'. The unitary transformation $e^{i\varepsilon S}$ is then the S-matrix or the evolution operator. It can also be written as $e^{iH_0 t/\hbar} e^{iH(t-t_0)/\hbar} e^{-iH_0 t_0}$ for $\varepsilon = 1$, and reflects the symmetry under time translation.