Journal of Theoretical Physics

Founded and Edited by M. Apostol **158** (2007)

ISSN 1453-4428

A power-law spectral line

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Abstract

A power law is derived for the tail of a spectral line. It is shown that there exists an excess of radiation for high frequencies due to the short-time correlations between the radiating atoms.

The intensity of a spectral line is basically given by

$$I(\omega) \sim \int d\tau \cdot e^{i\omega\tau}\varphi(\tau) \tag{1}$$

(the real part) where $\varphi(\tau)$ is a correlation function,

$$\varphi(\tau) \sim \int dt \cdot \mu(t+\tau)\mu(t) \ , \tag{2}$$

and μ is a dipole momentum. ω in equation (1) is in fact $\omega - \omega_0$, where ω_0 is the frequency peak. Collisions give rise to an exponential factor of the form

$$\mu(t) \sim e^{i\sum \eta_i(t)} \tag{3}$$

where $\eta_i$are phase shifts, such as

$$\mu(t+\tau)\mu(t) \sim e^{i\sum_\tau \eta_i} \tag{4}$$

contains the phase shifts in the time interval τ. The phase shifts η_i are viewed as stochastic variables.[1]

For long times τ we may write

$$\varphi(\tau + d\tau) - \varphi(\tau) = d\varphi(\tau) = nv\sigma d\tau \left\langle e^{i\eta} - 1 \right\rangle \varphi(\tau) \tag{5}$$

where n is the particle density, v is their thermal velocity, σ is the cross-section and the average is taken over one collision. This average can be denoted by $-\alpha$, so

$$\varphi(\tau) = e^{-nv\sigma\alpha\tau} = e^{-a\tau} \ . \tag{6}$$

From (1) we get the Lorentz shape

$$I(\omega) \sim \frac{a_1}{(\omega + a_2)^2 + a_1^2} \tag{7}$$

[1]Seee for instance P. W. Anderson, Phys. Rev. 76 647 (1949); *ibid*, **86** 809 (1952).

of the spectral line, where $a = a_1 + ia_2$. It contains both a frequency shift $\Delta\omega = a_2$ and a line breadth $\delta\omega = a_1$. For $\omega_0 \to 0$ the matrix elements of the dipole which give the transitions are quasi-clasical and α is real ($a = a_1$); we get the Debye shape of the spectral line

$$I \sim \frac{a}{\omega^2 + a^2} \ . \tag{8}$$

It contains only a line breadth $\delta\omega = a$.

We wish to estimate approximately the correlation function in the limit $\tau \to 0$, which is relevant for the tail $\omega \to \infty$ of the line shape.

In this case (4) gives

$$\begin{aligned} \mu(t+\tau)\mu(t) \sim \left\langle e^{i\sum_\tau \eta} \right\rangle = \exp\left[-\eta^2(\tau/2\Delta t)\right] \simeq \\ \simeq \exp\left[-\eta^2 - \eta^2 \ln(\tau/2\Delta t)\right] \end{aligned} \tag{9}$$

for $\tau \sim \Delta t \to 0$, where η^2 is an average and Δt is the mean time between two successive collisions. Equation (9) for $\tau \leq \Delta t$ gives a correlation function

$$\varphi(\tau) \simeq \exp\left[\eta^2 \ln(\tau/2\Delta t)\right] \tag{10}$$

for $\tau \geq \Delta t$, which makes sense in the t-integration in (2) and τ-integration in (1). We may therefore take an interpolation

$$\varphi(\tau) \simeq \exp\left[\eta^2 \ln(\tau/2\Delta t) - a\tau\right] \tag{11}$$

for the correlation function, which combines a power law $(\tau/2\Delta t)^{\eta^2}$ for small τ with an exponential tail for large τ, for small values of the parameter η^2.

The intensity given by (1) is estimated with this correlation function by the steepest descent. We get

$$I(\omega) \sim \eta(\eta^2/\Delta t)^{\eta^2} \frac{1}{(\omega^2 + a^2)^{1/2+\eta^2/2}} \ . \tag{12}$$

This is a qualitative difference with respect to (7) or (8). The former go like $1/\omega^2$ in the limit $\omega \to \infty$, while the latter goes like $1/\omega^{1+\eta^2} \sim 1/\omega$ in the same limit. The excess of radiation comes from short-time correlations between the radiating atoms.