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Specific heat of charged fermions in magnetic field

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Abstract

The leading contributions to the specific heat of an ensemble of charged fermions placed in a homogeneous magnetic field are computed.

An ensemble of N identical fermions, each of mass m, charge q and spin S, is placed in a homogeneous magnetic field H, directed along the z-axis. The single-particle energy is given by

$$E_{ns}(k) = \hbar\omega(n+1/2) + \hbar^2 k^2/2m - \alpha_s \ \ , \ \ n = 0, 1, 2, ... \ \ , \tag{1}$$

where $\omega = qH/mc$ is the cyclotron frequency, k is the vawevector along the z-axis and $\alpha_s = g_m \mu_B sH$ is the Zeeman energy; g_m is the gyromagnetic factor, μ_B is the Bohr magneton and $s = -S, -S+1, ...S$ is the spin projection along the z-axis.

The well-known specific heat of an ensemble of fermions in three dimensions,

$$c = \frac{\pi^2}{2}\frac{T}{\varepsilon_F} + ... \ \ , \tag{2}$$

where T is the temperature and ε_F is the Fermi energy, is extended in the present paper to

$$c = \frac{\pi^2}{2}\left(1 - f^2/12\varepsilon_F^2\right)\frac{T}{\varepsilon_F} + ... \ \ , \tag{3}$$

where

$$f^2 = S(S+1)(g_m\mu_B H)^2 - (\hbar\omega/2)^2 \ \ . \tag{4}$$

For electrons $q = -e$, $S = 1/2$, $g_m \cong 2$ and we get $f^2 = 2(\mu_B H)^2$; we recall that the Bohr magneton is $\mu_B = e\hbar/2mc$, $\hbar$ is Planck's constant, c is the light speed and $-e$ is the electron charge.

In the following the computations leading to (3) are described.

The ensemble of fermions is confined to a rectangular box of sides $L_{x,y,z}$, whose volume is $V = L_x L_y L_z$. As it is well-known,[1] the motion of a charged particle in a homogeneous magnetic field is described by a harmonic oscillator along, say, the y-axis, shifted by $y_0 = -(c/qH)p_x$, where p_x is the momentum along the x-axis. An orbital degeneracy occurs, given by

$$N_{xy} = \frac{L_y}{\Delta y_0} = L_y / \frac{c\hbar}{qH}\frac{2\pi}{L_x} = L_x L_y \cdot \frac{qH}{ch} \quad , \tag{5}$$

so that the number of particles can be written as

$$N = V\frac{qH}{2\pi ch}\sum_s \sum_n \int_{-\infty}^{\infty} dk \frac{1}{\exp\left[E_{ns}(k) - \mu\right]\beta + 1} \quad , \tag{6}$$

where the energy $E_{ns}(k)$ is given by (1), μ is the chemical potential and $\beta = 1/T$ is the inverse of the temperature. The energy $\hbar\omega$ is much smaller than the temperature, $\hbar\omega\beta \ll 1$; for electrons, for instance, $\hbar\omega \sim 1K$ for a magnetic field $H = 1Ts$. Consequently, we may replace the summation over n in (6) by an integral, according to the formula

$$\sum_a^{b-1} f(n + 1/2) = \int_a^b dn \cdot f(n) - \frac{1}{24} f^{'}(n) \mid_a^b \quad . \tag{7}$$

This formula is valid for $|f(n + 1/2) - f(n - 1/2)| \ll |f(n)|$, and the Fermi distribution in (6) fulfils this condition, providing $\hbar\omega\beta \ll 1$. When applying (7) to the Fermi distribution in (6) the derivative-term in (7) gives contributions of order H^2, and we keep all the subsequent computations up to terms of this order. Doing so, and after straightforward manipulations, the number of particles is re-expressed as

$$N = \sum_s N_s^b - \frac{1}{12} \cdot g\frac{V}{(2\pi)^2} \cdot m\omega^2 \cdot \frac{\partial}{\partial\mu} A \quad , \tag{8}$$

where

$$N_s^b = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\varepsilon \cdot \varepsilon^{1/2} \frac{1}{\exp\left(\varepsilon - \mu_s\right)\beta + 1} \tag{9}$$

is the bulk contribution to the number of particles (per spin), with the reduced chemical potential $\mu_s = \mu + \alpha_s$,

$$A = \frac{1}{2}\left(\frac{2m}{\hbar^2}\right)^{1/2} \int_0^\infty d\varepsilon \cdot \varepsilon^{-1/2} \frac{1}{\exp\left(\varepsilon - \mu\right)\beta + 1} \quad , \tag{10}$$

and $g = 2S+1$ is the spin degeneracy. The Fermi integrals appearing in (9) and (10) are estimated according to the well-known formula

$$\begin{gathered} \int_0^\infty d\varepsilon \cdot f(\varepsilon) \tfrac{1}{\exp(\varepsilon - \mu)\beta + 1} = \int_0^\mu d\varepsilon \cdot f(\varepsilon) + \tfrac{\pi^2 T^2}{6} f^{'}(\mu) + \\ + \tfrac{7\pi^4 T^4}{360} f^{'''}(\mu) + \ldots \quad , \end{gathered} \tag{11}$$

valid for $\mu\beta \gg 1$. As we shall see later, one needs to go in (11) up to terms of order T^4, at least. The Zeeman energy α_s is much smaller than the temperature, $\alpha_s\beta \ll 1$; for electrons, for example, the Zeeman splitting is $\sim 1K$ for a magnetic field $H = 1Ts$. Consequently, we expand N_s^b given by (9) in powers of $\alpha_s\beta$ up to the second order, and thereafter perform the summation over s. In doing this we denote

$$\sum_s \alpha_s^2 = g\alpha^2 \quad , \tag{12}$$

where $\alpha^2 = (1/3)S(S+1)(g_m\mu_B H)^2$. The number of particles given by (8) can now be written as

$$N = g\frac{V}{6\pi^2}\left(\frac{2mT}{\hbar^2}\right)^{3/2}(\ln z)^{3/2}\cdot \\ \cdot\left[1+\frac{1}{8}\left(\pi^2+\rho\right)\frac{1}{(\ln z)^2}+\frac{\pi^2}{64}\left(\frac{7\pi^2}{10}+\rho\right)\frac{1}{(\ln z)^4}+\frac{49\pi^4}{3072}\rho\frac{1}{(\ln z)^6}\right] \quad , \tag{13}$$

where

$$\rho = 3(\alpha\beta)^2 - (\hbar\omega\beta/2)^2 \quad , \tag{14}$$

and the fugacity $z = \exp(\mu\beta)$ has been introduced. Now we can see that the correction H^2-term contains $(H\beta)^2$, and this is why we have to go up to T^4-terms in the expansion of the Fermi integrals. Introducing the Fermi energy $\varepsilon_F = \hbar^2 k_F^2/2m$, where the Fermi wavevector is given by $N = gVk_F^3/6\pi^2$, after some tedious algebra we obtain from (13)

$$\ln z = (\beta\varepsilon_F)\cdot\left[\begin{array}{c} 1-\frac{1}{12}\left(\pi^2+\rho\right)\frac{1}{(\beta\varepsilon_F)^2}-\frac{\pi^2}{48}\left(\frac{3\pi^2}{5}+\rho\right)\frac{1}{(\beta\varepsilon_F)^4}- \\ -\frac{\pi^4}{4608}\left(49+\frac{581\pi^2}{45}+\frac{497}{15}\rho\right)\frac{1}{(\beta\varepsilon_F)^6}\end{array}\right] . \tag{15}$$

for $\beta\varepsilon_F \gg 1$.

In a similar manner, and within the same approximations, the energy can be expressed as

$$E = \sum_s E_s^b + \frac{1}{24}\cdot g\frac{V}{(2\pi)^2}\cdot m\omega^2\cdot A - \sum_s \alpha_s N_s^b \quad , \tag{16}$$

where

$$E_s^b = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\varepsilon\cdot\varepsilon^{3/2}\frac{1}{\exp\left(\varepsilon-\mu_s\right)\beta+1} \tag{17}$$

is the bulk contribution per spin. The last term in (16) is

$$\sum_s \alpha_s N_s^b = \frac{1}{\beta}\cdot g\frac{V}{4\pi^2}\left(\frac{2mT}{\hbar^2}\right)^{3/2}(\ln z)^{1/2}\cdot \\ \cdot\left[1-\frac{1}{24}\frac{1}{(\ln z)^2}-\frac{7\pi^4}{384}\frac{1}{(\ln z)^4}\right](\alpha\beta)^2 \quad , \tag{18}$$

and the energy is given by

$$\beta E = g\frac{V}{10\pi^2}\left(\frac{2mT}{\hbar^2}\right)^{3/2}(\ln z)^{5/2}\cdot \\ \cdot\left[1+\frac{5}{8}\left(\pi^2-\frac{1}{3}\rho\right)\frac{1}{(\ln z)^2}-\frac{\pi^2}{192}\left(\frac{7\pi^2}{2}-\frac{5}{3}\rho\right)\frac{1}{(\ln z)^4}+\frac{35\pi^4}{9216}\rho\frac{1}{(\ln z)^6}\right] \quad . \tag{19}$$

From (13) and (19) we obtain also

$$\beta E/N = \frac{3}{5}(\ln z)\cdot\left[1+\left(\frac{\pi^2}{2}-\frac{1}{3}\rho\right)\frac{1}{(\ln z)^2}-\frac{\pi^2}{24}\left(\frac{11\pi^2}{5}+\frac{2}{3}\rho\right)\frac{1}{(\ln z)^4}\right] \tag{20}$$

whence, by using (15),

$$E/N = \frac{3}{5}\varepsilon_F + \frac{1}{4}\left(\pi^2-\rho\right)\frac{T^2}{\varepsilon_F} - \frac{3\pi^2}{80}\left(\pi^2+\frac{5}{9}\rho\right)\frac{T^4}{\varepsilon_F^3} \quad . \tag{21}$$

Since $\rho = (f\beta)^2$, with $f^2 = 3\alpha^2 - (\hbar\omega/2)^2$ as given by (4), we can recast the energy per particle (21) as

$$E/N = \frac{3}{5}\varepsilon_F + \frac{\pi^2}{4}\frac{T^2}{\varepsilon_F} - \frac{3\pi^4}{80}\frac{T^4}{\varepsilon_F^3} - \frac{f^2}{4\varepsilon_F}\left(1 + \frac{\pi^2}{12}\frac{T^2}{\varepsilon_F^2}\right) \quad , \tag{22}$$

whence the specific heat for constant volume is

$$c_v = \frac{\pi^2}{2}\left(1 - f^2/12\varepsilon_F^2\right)\frac{T}{\varepsilon_F} - \frac{3\pi^4}{20}\left(\frac{T}{\varepsilon_F}\right)^3 \quad . \tag{23}$$

From (22) we cam also obtain the magnetization, and see that the Landau diamagnetism is 1/3 of the Pauli paramagnetism.

The grand-canonical potential $\Omega = -pV = -(1/\beta)\ln Q$, where p is the pressure and Q is the grand-partition function, is defined by

$$\beta\Omega = -V\frac{qH}{2\pi ch}\sum_s\sum_n\int_{-\infty}^{\infty} dk \cdot \ln\left[1 + \exp\left[E_{ns}(k) - \mu\right]\beta\right] \quad , \tag{24}$$

where the energy $E_{ns}(k)$ is given by (1). By similar transformations it can be re-expressed as

$$\Omega = -\frac{2}{3}\sum_s E_s^b + \frac{1}{12}\cdot g\frac{V}{(2\pi)^2}\cdot m\omega^2\cdot A \quad . \tag{25}$$

On the other hand, by taking the derivative with respect to β in (24), we obtain

$$\Omega + \beta\frac{\partial\Omega}{\partial\beta} = E - \mu N \quad , \tag{26}$$

as expected, whence the entropy

$$S = \beta^2\frac{\partial\Omega}{\partial\beta} = \frac{5}{3}\beta E - N\ln z - \frac{1}{9}\cdot g\frac{V}{(2\pi)^2}\cdot m\omega^2\beta\cdot A + \frac{2}{3}\beta\sum_s \alpha_s N_s^b \quad . \tag{27}$$

Using (18) and (21), as well as the estimation of A given by (10) and (11), we can check easily that $S \sim T$ for $T \to 0$, so that the specific heats both for constant volume and for constant pressure coincide up to terms of order T^3, and are given by the first two terms in (23), as indicated in (3). The correction $f^2/12\varepsilon_F^2$ to the linear slope of c *vs* T given in (3) is extremely small; for the typical values $f \sim 1K$ (corresponding to $H = 1Ts$) and $\varepsilon_F \sim 10^5 K$ we get an extremely small contribution $f^2/12\varepsilon_F^2 \sim 10^{-11}$.

The effects of the interaction between fermions are included in the specific heat by replacing the mass m by an effective mass m^*, as usually in the theory of the Fermi liquid.

Similar calculations for two-dimensional fermions, whose specific heat is

$$c = \frac{\pi^2}{3}\frac{T}{\varepsilon_F} + \ldots \quad , \tag{28}$$

indicate that corrections due to the magnetic field are exponentially small in $(\beta\varepsilon_F)$.

References

[1] L. Landau, Z. Phys. **64** 629 (1930).