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**$d$-wave in high-$T_c$ superconductivity**

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

**Abstract**

It is shown that $d$-wave superconductivity may arise in high-$T_c$ superconducting cuprates. The critical temperature is estimated in a simple model for the BCS gap equation.

Recently, the well-known Bardeen-Cooper-Schrieffer (BCS) gap equation

$$\Delta(\mathbf{k})+\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')\frac{\tanh\beta E(\mathbf{k}')/2}{2E(\mathbf{k}')}\Delta(\mathbf{k}')=0 \tag{1}$$

has been brought back into discussion,[1] in connection with its integral equation properties. In equation (1) $\Delta(\mathbf{k})$ is the gap in the spectrum of the Cooper pairs, $V(\mathbf{k},\mathbf{k}')$ is the pair interaction, $\beta=1/T$ is the inverse of the temperature $T$ and

$$E(\mathbf{k})=\sqrt{\varepsilon^2(\mathbf{k})+\left|\Delta(\mathbf{k})\right|^2} \tag{2}$$

is the pair excitation spectrum, where $\varepsilon(\mathbf{k})=e(\mathbf{k})-\mu$ is the excitation spectrum of the free particles; $e(\mathbf{k})$ is their (band) energy and $\mu$ is the chemical potential. The critical temperature equation

$$\Delta(\mathbf{k})+\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')\frac{\tanh\beta_c\varepsilon'/2}{2\left|\varepsilon'\right|}\Delta(\mathbf{k}')=0 \tag{3}$$

is derived from equation (1) for the critical temperature $T_c=1/\beta_c$ where the gap vanishes. As it is well-known, these BCS equations are relevant for classical superconductivity of metals and alloys, superfluidity of $He^3$ and, as it is widely agreed, for the high-$T_c$ supercondyctivity.[2]

It has been noticed recently[1] that equation (3) is a Fredholm integral equation whose kernel may be viewed as a matrix, and, as such, it may have solutions even for repulsive interaction, in contrast with the current belief; the solution is the lowest negative eigenvalue of the kernel, which, in general, may exist even for an indefinite matrix. The only difficulty in getting this eigenvalue is that we have often to resort to numerical computations, and the kernel matrix is in general a matrix with labels in two- or three-dimensional $\mathbf{k}$-space. We present here an approximate analytical solution for high-$T_c$ superconductivity, by taking advantage of the low filling factor of the holes in the two-dimensional Fermi sea of the high-$T_c$superconducting cuprates. This idea was put forward in a previous paper.[3] For more information about high-$T_c$ superconductivity we refer the reader to Ref. 2. By using such a method we are able to get a $d$-wave superconductivity for an attractive interaction.

We start with the excitation spectrum

$$\varepsilon(\mathbf{k}) = -2t\left(\cos k_x + \cos k_y\right) - \mu \ , \tag{4}$$

where $t$ is the bandwidth parameter of a tetragonal cell. For low filling it is reasonable to expand $\varepsilon(\mathbf{k})$ in powers of $k$; we limit this expansion to the fourth order,

$$\varepsilon(\mathbf{k}) = -4t - \mu + tk^2 - \frac{1}{48}tk^4\left(3 + \cos 4\varphi\right) \ , \tag{5}$$

where $k_x = k\cos\varphi$ and $k_y = k\sin\varphi$. From this equation we get $k(\varepsilon, \varphi)$ as given by

$$\begin{aligned} \varepsilon &= -4t - \mu + tk_0^2 \ , \\ k^2 &= k_0^2 + \tfrac{1}{48}k_0^4\left(3 + \cos 4\varphi\right) \ . \end{aligned} \tag{6}$$

The Fermi surface is defined by $\varepsilon = 0$, *i.e.* $k_{0F}^2 = 4 + \mu/t$, and the filling factor $n$ is given by $n = k_{F0}^2\left(1 + k_{F0}^2/16\right)/2\pi$, or $k_{F0}^2 \simeq 2\pi n$.

We pass from the integration over $k'$ in equation (3) into integration over $\varepsilon'$, with the density of states given by

$$\rho d\varepsilon' = 2tk'dk' = \left[1 + \frac{1}{24}k_0^2(3 + \cos\varphi')\right]d\varepsilon' \ , \tag{7}$$

by using equations (6). We expand the interaction $V(\mathbf{k}, \mathbf{k}')$ in powers of $k, k'$ up to the fourth order, and express it as a function of $\varepsilon, \varepsilon', \varphi, \varphi'$ by means of equations (6). In addition, due to the low filling factor, it is reasonable to take $\varepsilon = \varepsilon' = 0$ in $V(\mathbf{k}, \mathbf{k}')$ and $\rho$, such that $k_0 = k_{F0}$. Equation (3) can then be written as

$$\Delta(\varphi) + I\int d\varphi' U(\varphi, \varphi')\Delta(\varphi') = 0 \ , \tag{8}$$

where

$$I = \frac{1}{16\pi^2 t}\int_{-\varepsilon_c}^{\varepsilon_c} d\varepsilon \frac{\tan\beta_c\varepsilon/2}{|\varepsilon|} \tag{9}$$

and $U = \rho V$; the integration in equation (9) is performed as usually with a cutoff $\varepsilon_c$.

We pass now to the interaction $V(\mathbf{k}, \mathbf{k}')$. We assume it consists of a on-site Coulomb repulsion of strength $v_0$ and an inter-site attraction of strength $v$ between nearest neighbours. The latter comes typically from terms like

$$c^+_{\mathbf{k}_1}c^+_{\mathbf{k}_2}c_{\mathbf{k}_3}c_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3}e^{i(\mathbf{k}_1-\mathbf{k}_3)\mathbf{r}_{ij}} \ , \tag{10}$$

where $c_{\mathbf{k}}(c^+_{\mathbf{k}})$ are annihilation (creation) fermion operators and $\mathbf{r}_{ij}$ is the position vector of the nearest neighbours. We take $\mathbf{k}_2 = -\mathbf{k}_1 = \mathbf{k}$ as usually (spin labels included), and the gap $\Delta(\mathbf{k}) = \left\langle c^+_{\mathbf{k}}c^+_{-\mathbf{k}}\right\rangle$. It is reasonable to assume that the gap in invariant to spatial reflection, so, beside factor $e^{i(\mathbf{k}-\mathbf{k}')\mathbf{r}_{ij}}$ we have also the factor $e^{i(\mathbf{k}+\mathbf{k}')\mathbf{r}_{ij}}$ in equation (10). Therefore, the pair interaction is proportional to $\cos k_x \cos k'_x + \cos k_y \cos k'_y$. Consequently, we take the interaction as[3]

$$V(\mathbf{k}, \mathbf{k}') = v_0 + v\left(\cos k_x \cos k'_x + \cos k_y \cos k'_y\right) \ . \tag{11}$$

According to the above discussion, its expansion is

$$V(\varphi, \varphi') = v_0 + v\left(2 - k_0^2 + k_0^4/8\right) + \frac{1}{8}vk_0^4 \cos 2\varphi \cos 2\varphi' \ , \tag{12}$$

and, by making use of equation (7),

$$U(\varphi,\varphi') = (v_0+2v) + \frac{1}{8}k_0^2(v_0-6v) + \frac{1}{8}vk_0^4\cos 2\varphi\cos 2\varphi' + \frac{1}{24}k_0^2\left[v_0+v(2-k_0^2)\right]\cos 4\varphi' \ . \quad (13)$$

With the definition

$$\Delta_n = \int d\varphi \cos n\varphi \Delta(\varphi) \ , \quad (14)$$

equation (8) is transformed into a system of equations

$$\begin{gathered} \Delta_0 + 2\pi\left[(v_0+2v)+\tfrac{1}{8}k_0^2(v_0-6v)\right]I\Delta_0 + \tfrac{\pi}{12}k_0^2\left[v_0+v(2-k_0^2)\right]I\Delta_4 = 0 \ , \\ \Delta_2 + \tfrac{\pi}{8}vk_0^4 I\Delta_2 = 0 \end{gathered} \quad (15)$$

and $\Delta_4 = 0$.

The first equation (15) gives an $s$-wave superconductivity. Very likely, it is forbidden by the Coulomb interaction $v_0$ which overwhelms the attraction $2v$. In this case, we are left with the $d$-wave equation (15), which gives the critical temperature

$$T_c = 1.14\varepsilon_c e^{-64\pi t/|v|k_0^2} \quad (16)$$

(integral $I = [\ln(1.14\beta_c\varepsilon_c 0]/8\pi^2 t)$. For a sufficiently large interaction and cutoff energy $\varepsilon_c$ this critical temperature can be pretty high. For instance, we may take $\varepsilon_c \simeq t$, and get a critical temperature as high as $T_c = 100K$ for $v \simeq 4.8eV$ and $t \simeq 2.4meV$ with the filling factor $n = 0.15$.

It is worth notingthat we may start with equation (3) and the separable interaction given by equation (11). Defining

$$\begin{gathered} \Delta_0 = \sum_{\mathbf{k}} \tfrac{\tanh\beta_c\varepsilon/2}{|\varepsilon|}\Delta(\mathbf{k}) \ , \\ \Delta_1 = \sum_{\mathbf{k}} \cos k_x \tfrac{\tanh\beta_c\varepsilon/2}{|\varepsilon|}\Delta(\mathbf{k}) \ , \\ \Delta_2 = \sum_{\mathbf{k}} \cos k_y \tfrac{\tanh\beta_c\varepsilon/2}{|\varepsilon|}\Delta(\mathbf{k}) \ , \end{gathered} \quad (17)$$

equation (3) becomes

$$\Delta_0 + \frac{1}{2}v_0\Delta_0 + \frac{1}{2}v\cos k_x\Delta_1 + \frac{1}{2}v\cos k_y\Delta_2 = 0 \ , \quad (18)$$

which leads to

$$\begin{gathered} \Delta_0 + \tfrac{1}{2}v_0 I_0\Delta_0 + vI_1\Delta_1 = 0 \ , \\ \Delta_1 + \tfrac{1}{2}v_0 I_1\Delta_0 + \tfrac{1}{2}v\left(I_2+I_3\right)\Delta_1 = 0 \ , \end{gathered} \quad (19)$$

where

$$\begin{gathered} I_0 = \sum_{\mathbf{k}} \tfrac{\tanh\beta_c\varepsilon/2}{|\varepsilon|} \ , \\ I_1 = \sum_{\mathbf{k}} \cos k_x \tfrac{\tanh\beta_c\varepsilon/2}{|\varepsilon|} = \sum_{\mathbf{k}} \cos k_y \tfrac{\tanh\beta_c\varepsilon/2}{|\varepsilon|} \ , \\ I_2 = \sum_{\mathbf{k}} \cos^2 k_x \tfrac{\tanh\beta_c\varepsilon/2}{|\varepsilon|} = \sum_{\mathbf{k}} \cos^2 k_y \tfrac{\tanh\beta_c\varepsilon/2}{|\varepsilon|} \ , \\ I_3 = \sum_{\mathbf{k}} \cos k_x \cos k_y \tfrac{\tanh\beta_c\varepsilon/2}{|\varepsilon|} \ , \end{gathered} \quad (20)$$

and $\Delta_1 = \Delta_2$. The system of equations (19) has in principle two eigenvalues. It is difficult to get them, unless a numerical evaluation is made for the integrals $I_{0,1,2,3}$. An expansion in powers of the filling factor can be applied for their estimation. However, the solutions are not what we are looking for, because the angular symmetry is hidden in the definition of $\Delta_0, \Delta_1$. The expansion in angular variable as presented above is more appropriate.

The method presented here can be applied to other types of interactions and energy band structure.

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