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Attraction force between a polarizable point-like particle and a semi-infinite solid

M. Apostol and G. Vaman
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Abstract

The attraction force between a polarizable point-like particle and a semi-infinite solid is derived by computing the eigenmodes of the electromagnetic field interacting with matter. The calculations are based on the electromagnetic potentials and the equation of motion of the polarization, as in the elementary classical theory of dispersion. These two ingredients lead to coupled integral equations for polarization, which we solve. The force is computed from the zero-point energy (vacuum fluctuations) of the electromagnetic field interacting with matter. A d^{-4}-force is found in the non-retarded regime (Coulomb interaction), where d is the distance between the particle and the surface of the body. This force corresponds to the classical van der Waals-London interaction. It arises from the surface plasmons excited in the body. In the retarded regime there is no force between the particle and the semi-infinite solid, nor between any pair of particles. Such a negative result is due to the fact that we assume a classical dynamics for the point-like particle, which is valid in the non-retarded regime but does not hold anymore in the retarded one.

As it is well known, the zero-point energy (vacuum fluctuations) gives rise to an atttractive force between two polarizable pieces of matter.[1]-[3] In the non-retarded limit (Coulomb interaction) this is the well-known van der Waals-London force; for a pair of point-like particles separated by distance R it goes like R^{-7}. For a pair of semi-infinite bodies (two halves of space) separated by distance d, the van der Waals-London force goes like d^{-3}. Originally, such an attractive force has been derived by Casimir in the retarded regime,[4] by estimating the eigenmodes of the electromagnetic field interacting with two ideal, perfectly reflecting semi-infinite metals separated by distance d; in this case the Casimir force goes like d^{-4}. A similar force ($\sim d^{-5}$) was also derived for an atom-metal couple, or for a pair of atoms ($\sim R^{-8}$).[5]

Recently we re-investigated this subject within our theory of the electromagnetic field interacting with polarizable matter.[6] This theory is based on the electromagnetic potentials and the equation of motion of the polarization, as in the elementary theory of classical dispesion. These two ingredients lead to coupled integral equations, whose eigenmodes spectrum was calculated for two semi-infinite bodies. It was shown that the van der Waals-London force arises from surface plasmons, while the Casimir force originates in the surface plasmon-polariton modes. We extend here these calculations to a point-like particle interacting with a semi-infinite body, where we assume a classical dynamics for the particle. We show that an attractive force appears in this case, which goes like d^{-4}, where d is the distance between the particle and the surface of the body. This force occurs in the non-retarded limit, it is due to the surface plasmons, and corresponds to the van der Waals interaction. The result can be checked directly by applying the theory to a pair of point-like particles. In the retarded regime there is no branch of roots for the dispersion equations of the eigenmodes, and, consequently, there is no such a force. Similarly, there is no such a force between any pair of classical point-like particles. This negative result, in contrast with Casimir's result, is due to the fact that in the retarded regime the classical dynamics is not valid anymore for the point-like particle, in contrast with the non-retarded regime.

We consider a point-like polarizable particle located at $\mathbf{R}_0 = (0, 0, -d)$ and a semi-infinite (half-space) solid extending over the region $z > 0$, with a free surface in the (x, y)-plane. We adopt a generic model of matter polarization, consisting of mobile elementary charges $-e$ and mass m, and describe their density disturbances by $\delta n = -ndiv\mathbf{u}$, where $\mathbf{u}$ is a displacement field and n (constant) is the particle concentration. Such a description is valid for wavelengths much longer than the diplacement field u. The charge density is $\rho = endiv\mathbf{u}$ and the current density is $\mathbf{j} = -en\dot{\mathbf{u}}$.

For the semi-infinite solid the displacement field is taken as $\mathbf{u} = (\mathbf{v}, w)\theta(z)$, where $\mathbf{v}$ is the in-plane (x, y)-component, w is the component along the z-axis and $\theta(z) = 1$ for $z > 0$, $\theta(z) = 0$ for $z < 0$ is the step function. The charge density is given by

$$\rho = endiv\mathbf{u} = en\left(div\mathbf{v} + \frac{\partial w}{\partial z}\right)\theta(z) + enw(z=0)\delta(z) \ , \tag{1}$$

where we can notice the (de)polarization charge occurring at the surface. Similarly, the current density can be written as $\mathbf{j} = -en(\dot{\mathbf{v}}, \dot{w})\theta(z)$. These charge and current densities give rise to an electric field $\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - grad\Phi$, where the well-known electromagnetic potentials are given by

$$\mathbf{A}(\mathbf{R}, t) = \frac{1}{c}\int d\mathbf{R}'\frac{\mathbf{j}(\mathbf{R}', t - |\mathbf{R} - \mathbf{R}'|/c)}{|\mathbf{R} - \mathbf{R}'|} \ , \ \Phi(\mathbf{R}, t) = \int d\mathbf{R}'\frac{\rho(\mathbf{R}', t - |\mathbf{R} - \mathbf{R}'|/c)}{|\mathbf{R} - \mathbf{R}'|} \ . \tag{2}$$

We use the notation $\mathbf{R} = (\mathbf{r}, z)$, Fourier representation

$$\mathbf{v}(\mathbf{r}, z; t) = \sum_{\mathbf{k}}\int d\omega \mathbf{v}(\mathbf{k}, z; \omega)e^{i\mathbf{kr}}e^{-i\omega t} \tag{3}$$

and a similar one for w, and the well-known Fourier transform

$$\frac{1}{\sqrt{r^2 + z^2}} = \frac{1}{A}\sum_{\mathbf{k}}\frac{2\pi}{k}e^{-k|z|}e^{i\mathbf{kr}} \tag{4}$$

for the Coulomb potential, where A is the in-plane cross-sectional area. When retardation is included, we use the Fourier transform[7]

$$\frac{e^{i\frac{\omega}{c}\sqrt{r^2+z^2}}}{\sqrt{r^2 + z^2}} = \frac{1}{A}\sum_{\mathbf{k}}\frac{2\pi i}{\kappa}e^{i\mathbf{kr}}e^{i\kappa|z|} \ , \tag{5}$$

where $\kappa = \sqrt{\omega^2/c^2 - k^2}$.

Similarly, we consider a small, polarizable particle of radius a and charge $-q$, located at $\mathbf{R}_0$, and represents its displacement field by $a^3\mathbf{u}_0\delta(\mathbf{R} - \mathbf{R}_0)$. Its charge density can then be written as

$$\rho_0 = q\left(\mathbf{u}_0 grad\right)\delta(\mathbf{R} - \mathbf{R}_0) \ , \tag{6}$$

and the current density is given by $\mathbf{j}_0 = -q\dot{\mathbf{u}}_0\delta(\mathbf{R} - \mathbf{R}_0)$. We can see that the total charge of the particle is zero, and we can also recognize in equation (6) the dipole momentum $-q\mathbf{u}_0$. The charge density ρ_0 and the current density $\mathbf{j}_0$ give rise to an electric field $\mathbf{E}_0$ which can be computed by using equation (2).

The displacement $\mathbf{u}$ obeys the equation of motion

$$m\ddot{\mathbf{u}} = -e\left(\mathbf{E} + \mathbf{E}_0\right) - m\omega_1^2\mathbf{u} \ , \tag{7}$$

where ω_1 is a parameter. Such an equation is well known in the elementary theory of dispersion, it being able to simulate a metallic plasma ($\omega_1 = 0$), or a dielectric. We leave aside the dissipation, a possible external field, and limit ourselves to non-relativistic motion. Similarly, we adopt for the displacement field $\mathbf{u}_0$ of the particle charge the equation of motion

$$m\ddot{\mathbf{u}}_0 = -q\mathbf{E}\left|_{\mathbf{R}=\mathbf{R}_0}\right. - m\omega_0^2\mathbf{u}_0 \ , \tag{8}$$

where ω_0 is another parameter (usually much greater than the characteristic electromagnetic frequencies).

In general, if $\mathbf{E}$ is the field which acts upon the particle charge we get

$$\mathbf{u}_0 = \frac{q}{m}\frac{1}{\omega^2 - \omega_0^2}\mathbf{E} \simeq -\frac{q}{m\omega_0^2}\mathbf{E} \tag{9}$$

from equation (8), by a temporal Fourier transform. It follows that the dipole momentum per unit volume is $-q\mathbf{u}_0 = (q^2/ma^3\omega_0^2)\mathbf{E}$, whence we get the particle polarization

$$\alpha = \frac{q^2}{ma^3\omega_0^2} = \frac{\omega_{p0}^2}{4\pi\omega_0^2} \ , \tag{10}$$

where $\omega_{p0} = \sqrt{4\pi q^2/ma^3}$ is the plasma frequency.

We compute the electric fields $\mathbf{E}$ and $\mathbf{E}_0$ by making use of the electromagnetic potentials (2) and the charge and current densities given by above. Then, we introduce these fields in the equations of motion (7) and (8), get coupled integral equations for the displacements $\mathbf{u}$ and $\mathbf{u}_0$, solve them and obtain their electromagnetic eigenmodes. The corresponding eigenfrequencies are thereafter used to compute the force acting between the charged particle and a semi-infinite body.

First, we do the calculations for the non-retarded case, where $\mathbf{E} = -grad\Phi$, $\mathbf{E}_0 = -grad\Phi_0$, Φ and Φ_0 being the Coulomb potentials created by charges ρ (equation (1)) and, respectively ρ_0 (equation (6)). Making use of the Fourier transforms given by equations (3) and (4), leaving aside the arguments $\mathbf{k}$, ω for simplicity and introducing the notations $v = \mathbf{kv}/k$, $v_0 = \mathbf{kv}_0/k$, equation (8) leads to

$$\left(\omega^2 - \omega_0^2\right)v_0 = \frac{1}{2}k\omega_i^2 e^{-kd}\int_0 dz'\left[v(z') - iw(z')\right]e^{-kz'} \tag{11}$$

and $w_0 = -iv_0$, where $\omega_i = \sqrt{4\pi neq/m}$. In the same manner, equation (7) gives

$$\left(\omega^2 - \omega_1^2 - \omega_p^2\right)v = -\frac{1}{2}\omega_p^2 v(0)e^{-kz} + \frac{\omega_i^2}{nA}kv_0 e^{-k(d+z)} \tag{12}$$

and $ikw = \frac{\partial v}{\partial z}$, where $\omega_p = \sqrt{4\pi ne^2/m}$ is the plasma frequency of the semi-infinite body and $v(0) = v(z = 0)$. Making use of this latter relation and integrating by parts, equation (11) becomes

$$\left(\omega^2 - \omega_0^2\right)v_0 = \frac{1}{2}\omega_i^2 v(0)e^{-kd} \ . \tag{13}$$

It is worth noting, according to equations (12) and (13), that the semi-infinite body and the point-like particle are coupled through the frequency ω_i, which can be written also as

$$\omega_i^2 = \left(na^3\right)^{1/2} \omega_p \omega_{p0} \ , \tag{14}$$

where ω_{p0} is given in equation (10). Without coupling, equation (12) gives the well-known bulk plasmon frequency $\omega^2 = \omega_1^2 + \omega_p^2$ ($v(0) = 0$) and the surface plasmons $\omega^2 = \omega_1^2 + \frac{1}{2}\omega_p^2$ (for $v = v(0)e^{-kz}$).

The coupled surface plasmons can be obtained by solving the system of equations (12) and (13). In the limit of large ω_0 the frequency of the surface plasmons is given by

$$\Omega^2 = \omega_1^2 + \frac{1}{2}\omega_p^2 - 2\pi\alpha\frac{a^3k}{A}\omega_p^2 e^{-2kd} \ , \tag{15}$$

where the polarizability α given by equation (10) has beee introduced. For a metallic plasma $\omega_1 = 0$ and we get the frequencies

$$\Omega = \frac{1}{\sqrt{2}}\omega_p\left(1 - 2\pi\alpha\frac{a^3k}{A}e^{-2kd}\right) \ ; \tag{16}$$

for a dielectric $\omega_1 \gg \omega_p$ and we get the frequencies

$$\Omega \simeq \omega_1\left(1 - 4\pi^2\alpha\alpha_1\frac{a^3k}{A}e^{-2kd}\right) \ , \tag{17}$$

where we have introduced the polarization $\alpha_1 = \omega_p^2/4\pi\omega_1^2$.

We compute the force by

$$F = \frac{\partial}{\partial d}\sum_{\mathbf{k}}\frac{1}{2}\hbar\Omega \ , \tag{18}$$

where we recognize the zero-point energy (vacuum fluctuations) of the surface plasmons. Although the temperature effects can easily be included, it is easy to see that they are irrelevant for realistic situations, so we leave them aside, as usually. Using the frequency given by equation (16) for a semi-infinite plasma we get

$$F = \frac{3\hbar\omega_p}{8\sqrt{2}} \cdot \frac{\alpha a^3}{d^4} \ . \tag{19}$$

Similarly, for a dielectric, making use of the frequencies given by equation (17), we get the force

$$F = \frac{3\pi\hbar\omega_1}{4} \cdot \frac{\alpha\alpha_1 a^3}{d^4} \ . \tag{20}$$

It is well known that a similar force, which goes like d^{-3}, there exists between two semi-infinite bodies separated by distance d. It gives a R^{-6}-interaction energy between any pair of atoms, where R is the inter-atomic distance. This is the well-known van der Waals-London interaction (force goes like R^{-7}). The same van der Waals-London force is implied in the present case. It can be checked directly, by applying the procedure described above for two point-like polarizable particles separated by distance d. For the same frequency ω_0 for both particles we get a force

$$F = \frac{15\hbar\omega_0}{4} \cdot \frac{\alpha_1\alpha_2 a^6}{d^7} \ , \tag{21}$$

where $\alpha_{1,2}$are the polarizabilities of the two particles, which is a van der Waals-London force.

We pass now to the retarded interaction, where we use the Fourier transform given by equation (5). We introduce the notations $v_1 = \mathbf{kv}/k$ and $v_2 = \mathbf{k}_\perp \mathbf{v}/k$, where $k_\perp$ is a vector perpendicular to $\mathbf{k}$ ($\mathbf{kk}_\perp = 0$) of the same magnitude k ($k_\perp = k$). We use similar notations for $v_{01,2}$. The electric fields are computed straightforwardly by equations (2). Then, we use the equations of motion (7) and (8) in order to get coupled integral equations for the displacements fields $\mathbf{u}$ and $\mathbf{u}_0$. It is worth noting in deriving these equations the non-intervertibility of the derivatives and the integrals, according to the identity

$$\frac{\partial}{\partial z}\int_0^\infty dz^{'} f(z^{'})\frac{\partial}{\partial z^{'}}e^{i\kappa\left|z-z^{'}\right|} = \kappa^2\int_0^\infty dz^{'} f(z^{'})e^{i\kappa\left|z-z^{'}\right|} - 2i\kappa f(z) \tag{22}$$

for any function $f(z)$, $z > 0$.It is due to the discontinuity in the derivative of the function $e^{i\kappa\left|z-z^{'}\right|}$ for $z = z^{'}$. From the equations of motion for the field $\mathbf{u}_0$ we get immediately $w_0 = \frac{k}{\kappa}v_{01}$. Similarly, from the equations of motion for the field $\mathbf{u}$ we get $w = \frac{ik}{\kappa'^2}\frac{\partial v_1}{\partial z}$, where

$$\kappa^{'} = \sqrt{\kappa^2 - \frac{\omega_p^2}{c^2}\cdot\frac{\omega^2}{\omega^2-\omega_1^2}}\ . \tag{23}$$

Therefore, we are left with equations in the unknowns $v_{1,2}$ and $v_{01,2}$. Leaving aside, as usualy, the arguments $\mathbf{k}$, ω we get the first set of integral equations

$$\begin{gathered}\left(\omega^2-\omega_1^2\right)v_2 = -\frac{i\omega_p^2\omega^2}{2c^2\kappa}\int_0 dz' v_2(z')e^{i\kappa|z-z'|} - \frac{i\omega_i^2\omega^2}{2nAc^2\kappa}v_{02}e^{i\kappa(z+d)}\ , \\ \left(\omega^2-\omega_0^2\right)v_{02} = -\frac{i\omega_i^2\omega^2}{2c^2\kappa}\int_0 dz' v_2(z')e^{i\kappa(d+z')}\ .\end{gathered} \tag{24}$$

Taking the second derivative of the first equation we get

$$\frac{\partial^2 v_2}{\partial z^2} + \kappa^{'2}v_2 = 0\ . \tag{25}$$

Looking, therefore for solutions of the form $v_2 = A_2 e^{i\kappa' z}$, where A_2 are undetermined amplitudes, we get the dispersion equation

$$\frac{\kappa^{'}+\kappa}{\kappa^{'}-\kappa}\cdot\frac{\kappa}{\kappa^2+k^2} + 2\pi i\alpha\frac{a^3}{A}e^{2i\kappa d} = 0\ . \tag{26}$$

Similarly, the second set of equations is given by

$$\begin{gathered}\left(\omega^2-\omega_1^2+\omega_p^2\frac{k^2}{\kappa'^2}\right)v_1 = -\frac{i\omega_p^2\kappa\left(\kappa'^2+k^2\right)}{\kappa'^2}\int_0 dz' v_1(z')e^{i\kappa|z-z'|} + \\ +\frac{\omega_p^2k^2}{2\kappa'^2}v_1(0)e^{i\kappa z} - \frac{i\omega_i^2\left(\kappa^2-k^2\right)}{2nA\kappa}v_{01}e^{i\kappa(z+d)}\ , \\ \left(\omega^2-\omega_0^2\right)v_{01} = -\frac{i\omega_i^2\kappa}{2}\int_0 dz'\left[v_1(z') + \frac{ik^2}{\kappa\kappa'^2}\frac{\partial v_1(z')}{\partial z'}\right]e^{i\kappa(d+z')}\ .\end{gathered} \tag{27}$$

It is easy to see that v_1 satisfies the same equation (25); solutions of the form $v_1 = A_1 e^{i\kappa' z}$ lead to the dispersion equation

$$\frac{\kappa^{'}+\kappa}{\kappa^{'}-\kappa}\cdot\frac{\kappa\kappa^{'}+k^2}{\kappa\kappa^{'}-k^2}\cdot\frac{\kappa}{\kappa^2-k^2} + 2\pi i\alpha\frac{a^3}{A}e^{2i\kappa d} = 0\ . \tag{28}$$

The dispersion equations (26) and (28) have not a branch of roots, which might become dense in the limit of large κd. Therefore, we conclude that there is no force in the retarded regime, between a polarizable point-like particle and a semi-infinite solid. The polarization strength of the classical point-like particle is too weak when retardation is included to give rise to such a force. The same negative result can be obtained within the present approach for a pair of point-like classical particles.

In conclusion, we may say that we have computed herein the spectrum of the eigenmodes of the electromagnetic field interacting with a semi-infinite body and a polarizable point-like particle located at the distance d from the surface of the body. We have evaluated the attraction force in this case, from the zero-point energy (vacuum fluctuations), and found that a van der Waals-London force arises from the excitation of the surface plasmons in the non-retarded regime; this force goes like d^{-4}. We found no such force when retardation is included, either between a paticle and a semi-infinite body or between a pair of particles. This result is due to the fact that we assume a classical dynamics for the point-like particles, which is valid in the non-retarded regime but does not hold anymore in the retarded one.

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