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The effect of the inhomogeneities on the propagation of elastic waves in isotropic bodies

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Abstract

A new method is introduced for estimating the effects of the inhomogeneities on the propagation of the elastic waves in isotropic bodies. The method is based on the Kirchhoff electromagnetic potentials. It is aplied here for estimating the effect of a static density inhomogeneity, either extended or localized, on the elastic waves propagating in an infinite, or a semi-infinite (half-space) body. For a semi-infinite body the method leads to coupled integral equations, which are solved. It is shown that such a density inhomogeneity may renormalize the waves velocity, or may even produce dispersive waves, depending on the geometry of the body and the spatial extension of the inhomogeneity. The method can be extended to other types of geometries or inhomogeneities, as, for instance, those occurring in the elastic constants.

The effect of the inhomogeneities on the propagation of the elastic waves in structures with special, restricted geometries has always enjoyed a great deal of interest.[1]-[12] Apart from its practical importance in engineering, the problem is particularly relevant for the effect the seismic waves may have on the Earth's surface.[13]-[18] The propagation of elastic waves in bodies with finite, special geometries, like, for instance, a semi-infinite space, poses certain technical problems. We present herein a new method of dealing with elastic waves in isotropic media, borrowed from electromagnetism. The method is based on Kirchhoff retarded potentials for the wave equation. In the present paper we analyze the change produced in the eigenfrequencies of the elastic modes by static density inhomogeneities of a certain spatial extent distributed in an infinite, or a semi-infinite (half-space) isotropic body.

As it is well known,[19] the propagation of the elastic waves in an isotropic body is governed by the equation of motion

$$\rho\ddot{\mathbf{u}} = \mu\Delta\mathbf{u} + (\lambda+\mu)\, graddiv\mathbf{u} \ , \tag{1}$$

where ρ is the body density, $\mathbf{u}$ is the field displacement and $\lambda\, , \mu$ are the Lame coefficients. We leave aside the external forces and write this equation in the form

$$\frac{1}{v_t^2}\ddot{\mathbf{u}} - \Delta\mathbf{u} = q\cdot grad\cdot div\mathbf{u} \ , \tag{2}$$

where $v_t = \sqrt{\mu/\rho}$ is the velocity of the transverse waves, $q = v_l^2/v_t^2 - 1$ and $v_l = \sqrt{(\lambda+2\mu)/\rho}$ is the velocity of the longitudinal waves. As it is well-known, for reasons of stability, the inequality $q > 1/3$ (actually $q > 1$ for real bodies) holds. A particular solution of equation (2) is given by the well-known Kirchhoff potential[20]

$$\mathbf{u}(\mathbf{R},t)=\frac{q}{4\pi}\int d\mathbf{R}'\frac{grad\cdot div\mathbf{u}(\mathbf{R}',t-|\mathbf{R}-\mathbf{R}'|/v_t)}{|\mathbf{R}-\mathbf{R}'|}\ . \tag{3}$$

Indeed, making use of Fourier transforms and using also the well-known integral

$$\int d\mathbf{R}\frac{e^{i\mathbf{KR}+i\omega R/v_t}}{R}=-\frac{4\pi v_t^2}{\omega^2-v_t^2K^2}\ , \tag{4}$$

we get the eigenvalue equation

$$\left(-\rho\omega^2+\mu K^2\right)\mathbf{u}=-\left(\lambda+\mu\right)(\mathbf{Ku})\mathbf{K}\ , \tag{5}$$

where ωdenotes the frequency and $\mathbf{K}$ is the wavevector. One can check immediately that equation (5) gives the well-known transverse and longitudinal elastic waves propagating in an infinite isotropic body.

For a semi-infinite body extending over the region $z>0$, with a free surface in the (x,y)-plane $z=0$, we use

$$\mathbf{u}\rightarrow\mathbf{u}\theta(z)=(\mathbf{v},u_3)\theta(z) \tag{6}$$

for the displacement field, where $\theta(z)=1$ for $z>0$, $\theta(z)=0$ for $z<0$ is the step function, $\mathbf{v}$ is the (x,y) in-plane component and u_3 is the normal-to-surface component of the displacement (directed along the z-coordinate). We use Fourier transforms of the type

$$\mathbf{u}(\mathbf{r},z;t)=\sum_{\mathbf{k}}\int d\omega\mathbf{u}(\mathbf{k}\omega;z)e^{i\mathbf{kr}}e^{-i\omega t}\ , \tag{7}$$

where $\mathbf{R}=(\mathbf{r},z)$. The divergence occurring in equation (3) can then be written as

$$div\mathbf{u}=\left(div\mathbf{v}+\frac{\partial u_3}{\partial z}\right)\theta(z)+u_3(0)\delta(z)\ , \tag{8}$$

where we can see the occurrence of the surface term $u_3(0)=u_3(z=0)$. The gradient can be computed similarly, bu using the Fourier transform given by equation (7).

We assume a certain region in the body, whose shape and extension is described by a function $g(\mathbf{r},z)$, where the density of the body is modified according to

$$\rho\rightarrow\rho+\rho g(\mathbf{r},z)\ . \tag{9}$$

We employ equation (9) for describing an inhomogeneity in the body. It is easy to see that this change in density introduces a new source term in equation (2), which can be written as

$$-\frac{1}{v_t^2}g(\mathbf{r},z)\ddot{\mathbf{u}}(\mathbf{r},z;t)=\sum_{\mathbf{k}}\int d\omega\frac{\omega^2}{v_t^2}\mathbf{h}(\mathbf{k}\omega;z)e^{i\mathbf{kr}}e^{-i\omega t}\ , \tag{10}$$

where

$$\mathbf{h}(\mathbf{k}\omega;z)=\sum_{\mathbf{k}_1}g(\mathbf{k}-\mathbf{k}_1,z)\mathbf{u}(\mathbf{k}_1\omega;z)\ . \tag{11}$$

Consequently, equation (3) becomes

$$\begin{aligned}\mathbf{u}(\mathbf{R},t)=\frac{q}{4\pi}\int d\mathbf{R}'\frac{graddiv\mathbf{u}(\mathbf{R}',t-|\mathbf{R}-\mathbf{R}'|/v_t)}{|\mathbf{R}-\mathbf{R}'|}-\\-\frac{1}{4\pi v_t^2}\int d\mathbf{R}'\frac{g(\mathbf{r}',z')\ddot{\mathbf{u}}(\mathbf{R}',t-|\mathbf{R}-\mathbf{R}'|/v_t)}{|\mathbf{R}-\mathbf{R}'|}\ .\end{aligned} \tag{12}$$

Making use of the representations given above, and after performing conveniently a few integrations by parts, equation (12) can be simplified appreciably. The intervening integrals can be performed straightforwardly. They reduce to the known integral[21]

$$\int_{|z|}^{\infty} dx J_0\left(k\sqrt{x^2-z^2}\right)e^{i\omega x/c}=\frac{i}{\kappa_0}e^{i\kappa_0|z|} \ , \tag{13}$$

where J_0 is the Bessel function of the first kind and zeroth order and

$$\kappa_0=\sqrt{\frac{\omega^2}{v_t^2}-k^2} \ . \tag{14}$$

We get the system of coupled integral equations

$$\begin{gathered}\mathbf{v}(\mathbf{k}\omega;z)=-\tfrac{iq\mathbf{k}}{2\kappa_0}\int_0 dz'\mathbf{kv}(\mathbf{k}\omega;z')e^{i\kappa_0|z-z'|}-\tfrac{q\mathbf{k}}{2\kappa_0}\tfrac{\partial}{\partial z}\int_0 dz'u_3(\mathbf{k}\omega;z')e^{i\kappa_0|z-z'|}+\\ +\tfrac{i\omega^2}{2v_t^2\kappa_0}\int_0 dz'\mathbf{h}_{\parallel}(\mathbf{k}\omega;z')e^{i\kappa_0|z-z'|}\end{gathered} \tag{15}$$

and

$$\begin{gathered}u_3(\mathbf{k}\omega;z)=-\tfrac{q}{2\kappa_0}\tfrac{\partial}{\partial z}\int_0 dz'\mathbf{kv}(\mathbf{k}\omega;z')e^{i\kappa|z-z'|}+\tfrac{iq}{2\kappa_0}\tfrac{\partial^2}{\partial z^2}\int_0 dz'u_3(\mathbf{k}\omega;z')e^{i\kappa|z-z'|}+\\ +\tfrac{i\omega^2}{2v_t^2\kappa_0}\int_0 dz'h_3(\mathbf{k}\omega;z')e^{i\kappa_0|z-z'|} \ ,\end{gathered} \tag{16}$$

where $\mathbf{h}_{\parallel}$ is the in-plane component of the vector $\mathbf{h}$ defined by equation (11) and h_3 is its compoennt along the z-direction.

It is convenient to introduce the notations $v_1=\mathbf{kv}/k$, $v_2=\mathbf{k}_{\perp}\mathbf{v}/k$, and similar ones for the vector $\mathbf{h}$, where $\mathbf{k}_{\perp}$ is a vector perpendicular to $\mathbf{k}$, $\mathbf{kk}_{\perp}=0$, and of the same magnitude k. Under these conditions equation (15) for $v_2$reduces to

$$v_2(\mathbf{k}\omega;z)=\frac{i\omega^2}{2v_t^2\kappa_0}\int_0 dz'h_2(\mathbf{k}\omega;z')e^{i\kappa_0|z-z'|} \ . \tag{17}$$

This equation corresponds to the transverse wave polarized perpendicular to the plane of propagation (it is known in electromagnetism as the s-wave, from the German "senkrecht" which means "perpendicular"). Taking the second derivative with respect to z in this equation we get

$$\frac{\partial^2 v_2}{\partial z^2}=-\kappa_0^2 v_2-\frac{\omega^2}{v_t^2}h_2 \ . \tag{18}$$

Here, it is worth noting the non-invertibility of the (second) derivative and the integral in equation (17), as a result of the discontinuity in the derivative of the function $e^{i\kappa_0|z-z'|}$ for $z=z'$. In equation (18) we perform a Fourier transform with respect to the coordinate z. Introducing the wavevectors $\mathbf{K}=(\mathbf{k},\kappa)$ and $\mathbf{K}_1=(\mathbf{k},\kappa_1)$ and making use of equation (14), equation (18) becomes

$$\left(\frac{\omega^2}{v_t^2}-K^2\right)v_2(\mathbf{K}\omega)=-\frac{\omega^2}{v_t^2}\sum_{\mathbf{K}_1}g(\mathbf{K}-\mathbf{K}_1)v_2(\mathbf{K}_1\omega) \ . \tag{19}$$

We assume first that function $g(\mathbf{R})$ is a constant, $g(\mathbf{R})=g$. Then, $g(\mathbf{K})=g\delta_{\mathbf{K},0}$ and equation (19) gives the frequency

$$\omega=\frac{v_t}{\sqrt{1+g}}K \ , \tag{20}$$

which shows that the wave velocity is renormalized as a consequence of the change in density, as described by the parameter g. Second, we assume that the function $g(\mathbf{R})$ is localized at some position $\mathbf{R}_0$ in the body over a small spatial range of linear extension a. Then, its Fourier transform can be taken almost constant, $g(\mathbf{K})\simeq ga^3/V$, over a range $\sim 1/a$, where V is the volume of the body and $g = g(\mathbf{R}_0)$. Under these conditions we get from equation (19) the dispersion relation

$$1 = -\frac{\omega^2 ga^3}{v_t^2 V}\sum_{\mathbf{K}}\frac{1}{\omega^2/v_t^2 - K^2} \ . \tag{21}$$

For small values of g the solutions of this equation are given by

$$\omega^2/v_t^2 = K^2 - \frac{g\omega^2}{6\pi^2 v_t^2} = K^2 - \frac{g}{6\pi^2}K^2 + ..., \tag{22}$$

whence, in the first approximation, we get another renormalizaton of the wave velocity

$$v_t \rightarrow v_t\left(1 - \frac{g}{12\pi^2}\right) \ . \tag{23}$$

We note that this renormalization does not depend on the spatial extension of the function $g(\mathbf{R})$. We also note that these results are the same for an infinite body. For a general function $g(\mathbf{R})$ we obtains a renormalization of the velocity comprised between the two limiting cases given above by equations (20) and (23). We can also consider a layer of thickness a, *i.e.* take $g(\mathbf{R}) = g(z - z_0)$ and $g(\mathbf{k}, \kappa) \simeq (ga/L)\,\delta_{\mathbf{k},0}$, where L is the spatial extension of the body along the z-direction and $g(\mathbf{k}, \kappa)$ extends over a range $\sim 1/a$ as a function of κ. The velocity is then renormalized according to

$$v_t \rightarrow v_t\left(1 - \frac{g}{4\pi}\right) \ . \tag{24}$$

We turn now to equation (15) written for v_1 and equation (16) for u_3. We leave aside arguments $\mathbf{k}\,,\omega$ for simplicity, and preserve explicitly only the argument z. It is easy to see that these two equations imply

$$u_3(z) = -\frac{i}{k}\frac{\partial v_1}{\partial z} - \frac{\omega^2}{2v_t^2\kappa_0 k}\frac{\partial H_1}{\partial z} + \frac{i\omega^2}{2v_t^2\kappa_0}H_3(z) \ , \tag{25}$$

where

$$H_{1,3}(z) = \int_0 dz' h_{1,3}(z')e^{i\kappa_0|z-z'|} \ . \tag{26}$$

We introduce $u_3(z)$ as given by equation (25) in equation (15) for $v_1(z)$ and take the second derivative in the resulting equation. We get

$$\frac{\partial^2 v_1}{\partial z^2} + \kappa_0^{'2} v_1 = \frac{i\omega^2}{2v_t^2\kappa_0}\left(\frac{\partial^2 H_1}{\partial z^2} + \frac{\kappa_0^2 v_t^2}{v_l^2}H_1\right) + \frac{qk\omega^2}{2v_l^2\kappa_0}\frac{\partial H_3}{\partial z} \ , \tag{27}$$

where

$$\kappa_0^{'} = \sqrt{\frac{\omega^2}{v_l^2} - k^2} \ . \tag{28}$$

We introduce Fourier transforms with respect to the z-coordinate both in equation (25) and equation (27). The Fourier transforms of the functions $H_{1,3}(z)$ are

$$H_{1,3}(\kappa) = -\frac{2i\kappa_0}{\kappa^2 - \kappa_0^2}h_{1,3}(\kappa) \tag{29}$$

for $\kappa \neq \kappa_0$. Restoring the arguments, $h_1(\kappa)$ is written, by equation (11), as

$$h_1(\mathbf{K}) = \sum_{\mathbf{K}_1} g(\mathbf{K}-\mathbf{K}_1)v_1(\mathbf{K}_1) \ ; \tag{30}$$

a similar expression holds for h_3. Doing so, we get two coupled equations

$$u_3(\mathbf{K}) - \frac{\kappa}{k}v_1(\mathbf{K}) + \frac{\omega^2}{\omega^2 - v_t^2K^2}\sum_{\mathbf{K}_1} g(\mathbf{K}-\mathbf{K}_1)\left[u_3(\mathbf{K}_1) - \frac{\kappa}{k}v_1(\mathbf{K}_1)\right] = 0 \tag{31}$$

and

$$\begin{gathered}(\omega^2 - v_t^2K^2)(\omega^2 - v_l^2K^2)v_1(\mathbf{K}) + \omega^2(\omega^2 - v_l^2\kappa^2 - v_t^2k^2)\textstyle\sum_{\mathbf{K}_1} g(\mathbf{K}-\mathbf{K}_1)v_1(\mathbf{K}_1) + \\ + qv_t^2\kappa k\omega^2 \textstyle\sum_{\mathbf{K}_1} g(\mathbf{K}-\mathbf{K}_1)u_3(\mathbf{K}_1) = 0 \ .\end{gathered} \tag{32}$$

In analyzing these equations we proceeed as before. For a constant function $g(\mathbf{R}) = g$, whose Fourier transform is $g(\mathbf{K}) = g\delta_{\mathbf{K},0}$, equations (31) and (32) give two types of waves. For the longitudinal wave, $u_3 = \kappa v_1/k$, equation (31) is satisfied identically, while from equation (32) we get a renormalization of the velocity v_l which is the same as that given above by equation (20). For the transverse wave $u_3 = -kv_1/\kappa$ (p-wave, whose polarization lies in the plane of propagation) we get from equations (31) and (32) the same renormalization of the velcity v_t as that given by equation (20).

We assume now a function $g(\mathbf{R})$ localized at some position $\mathbf{R}_0$ within the body and extending over a range $\sim a$. Its Fourier transform can be taken as $g(\mathbf{K}) \simeq ga^3/V$ for $\mathbf{K}$ extending over a range $\sim 1/a$ and $g = g(\mathbf{R}_0)$. It is easy to see that, according to equations (31) and (32), the velocity v_t is not renormalized in the first order of the (small) parameter g, but the velocity v_l acquires a renormalization given by

$$v_l \rightarrow v_l\left(1 - \frac{g}{36\pi^2}\right) \ . \tag{33}$$

Similarly, for a layer of thickness a the velocity v_t is not renormalized in the first order of the parameter g but the frequency of the longitudinal waves becomes

$$\omega = v_lK\left(1 - \frac{gak}{4}\right) \ ; \tag{34}$$

we can see that the longitudial waves become dispersive in this case.

For comparison we give here the results for a density inhomogeneity in an infinite elastic body. By using Fourier transforms, equation (12) leads to

$$\mathbf{u}(\mathbf{K}\omega) = \frac{qv_t^2}{\omega^2 - v_t^2K^2}(\mathbf{K}\mathbf{u})\mathbf{K} - \frac{\omega^2}{\omega^2 - v_t^2K^2}\mathbf{h}(\mathbf{K}\omega) \ , \tag{35}$$

where

$$\mathbf{h}(\mathbf{K}\omega) = \sum_{\mathbf{k}_1} g(\mathbf{K}-\mathbf{K}_1)\mathbf{u}(\mathbf{K}_1\omega) \tag{36}$$

and we have used the integral given by equation (4). Equation (35) reduces to

$$u_{1,2}(\mathbf{K}\omega) + \frac{\omega^2}{\omega^2 - v_{l,t}^2K^2}\sum_{\mathbf{K}_1} g(\mathbf{K}-\mathbf{K}_1)u_{1,2}(\mathbf{K}_1\omega) = 0 \tag{37}$$

for the longitudinal waves $u_1 = \mathbf{uK}/K$ (velocity v_l) and, respectively, transerse waves $u_2 = \mathbf{uK}_\perp/K$ (velocity v_t), where $\mathbf{K}_\perp$ is a vector perpendicular to the wavevector $\mathbf{K}$, $\mathbf{KK}_\perp = 0$, and of the same magnitude K. Both equations (37) lead to a dispersion equation of the same form as the one corresponding to the s-wave (equation (19)). For an extended inhomogeneity both $v_{t,l}$ are renormalized according to equation (20), for a localized inhomogeneity both velocities are renormalized according to equation (23). This is different than the semi-infinite body (compare with equation (33)).

In conclusion we may say that we have introduced herein a new method, based on the Kirchhoff electromagnetic potentials, to estimate the effects of density inhomogeneities on the propagation of the elastic waves in isotropic bodies. We have applied this method both to an infinite body and a semi-infinite (half-space) body. For an infinite body a density inhomogeneity renormalizes the velocity of the transverse and longitudinal waves. We have estimated this effect both for an extended and a localized inhomogeneity, or for a layer, assuming that the strength of the inhomegenity is small (parameetr g). For a semi-infinite body the present method leads to coupled integral equations which we have solved. The transverse s-wave is affected in the same manner as in an infinite body, and this holds also for all the waves for an extended inhomogeneity, as expected. For a localized inhomogeneity the transverse p-wave is affected in the second-order of the parameter g, while the longitudinal wave undergoes a renormalization of velocity (different than in an infinite body). In addition, for a layer inhomogeneity, the longitudinal waves become dispersive.

The method presented here can be extended to other types of inhomogeneities, as, for instance, those produced in the elastic properties of the body (the Lame coefficients). This problem is left for a forthcoming investigation.

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