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Plasmons, polaritons and diffraction of the electromagnetic field in cylindrical geometries

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Abstract

We report on some rigourous investigations regarding plasmons, polaritons and diffracted electromagnetic fields in cylindrical geometries. Some of the results are negative, or unphysical, implying ultraviolet divergencies. We examine this problem for a circular aperture in a two-dimensional sheet (a screen), an infinite cylindrical hole (a tunnel) and a finite cylindrical hole bored into a slab of finite thickness, and for their complementary structures, *i.e.* a disk and an infinite (finite) cylindrical rod. For some cases we get exact solutions, for others we are not able to get exact solutions. The method employed is based on solving coupled integral equations which follow from the equation of motion of matter polarization in an ideal model of jellium-like plasma and the Kirchhoff radiation formulae for the electromagnetic potentials. This method has been used by us recently for a semi-infinite plasma, a plasma slab and a sphere (Mie theory), where we get exact, physical solutions. First, we (re-)derive the plasmons for a two-dimensional plasma sheet and compute the reflected and transmitted field for a plane wave incident on the sheet. Next, we treat the circular aperture and the disk, where we encounter unphysical divergencies. For an infinite cylindrical hole and an infinite rod we get exact solutions for plasmons, which can be called "cylindrical" plasmons, and compute the dielectric response. For a finite cylindrical hole and a finite rod, as well as for other cases, we are not able to solve the coupled integral equations.

Key words: *plasmons; diffraction; circular cylindrical geometries*

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1 Introduction

The interest for the electromagnetic field interacting with matter in structures with special, restricted geometries[1]-[9] is motivated by the role played by plasmons and polaritons (including surface plasmons and surface plasmon-polariton modes) in the reflected, refracted, transmitted or diffracted field[10]-[32]. On the other hand, there has always been an interest in the status of the so-called "effective medium permittivity" theory in structures with special geometries. Recently, we have introduced a method for treating such problems, based on the equation of motion for matter polarization and the Kirchhoff radiation formulae for electromagnetic potentials[33]. Typically, this approach leads to coupled integral equations. By using this method we have derived the surface plasmons and the surface plasmon-polaritons in a semi-infinite (half-space) plasma and a plasma slab[34], together with the reflected, refracted and transmitted field, and obtained generalized Fresnel equations. Making use of the same method we re-derived Mie theory of diffraction by a metallic sphere[35], where we have stressed upon the role played by the "spherical" plasmons, and calculated van der Waals-London and Casimir forces acting between two semi-infinite bodies[36], as arising from the electromagnetic eigenmodes of matter interacting with the electromagnetic field. Interface plasmons have also been derived in rectangular geometries for various pairs of bodies (metals and dielectrics)[34].

We attempt herein to extend this method to bodies with cylindrical geometries, focusing in particular on the plasmon and polariton modes and the diffracted field. This problem has been tackled extensively in the past[37]-[44], within various approximations. In particular, diffraction of scalar fields by small holes or circular apertures in a metallic screen have been investigated by approximate methods[45, 46], under certain boundary conditions. First, we derive the plasmon modes for a two-dimensional plasma sheet (a screen), and compute the reflected and transmitted electromagnetic field for a plane wave incident on the sheet. Next, we show that unphysical ultraviolet divergencies occur for plasmons in a two-dimensional plasma sheet with a circular aperture, as well as for a disk. We were not able to solve the coupled integral equations for diffraction by a circular aperture (or a disk). We get exact solutions for "cylindrical" plasmons in an infinite cylindrical hole and an infinite cylindrical rod. For a circular cylindrical hole bored into a plasma slab of finite thickness, as well as for a finite cylindrical rod we were not able to solve the coupled integral equations. The calculations are performed for an ideal plasma, but they can be extended straightforwardly to a dielectric, loss included.

We adopt a simple model of an ideal jellium-like plasma consisting of mobile charges $-e$ with mass M and concentration n moving in a uniform, rigid, neutralizing background of positive charges. This model is a convenient representation for an ideal metal in the range of optical frequencies. We introduce a disturbance δn in the charge density given by $\delta n = -ndiv\mathbf{u}$, where $\mathbf{u}$ is a displacement field in the positions of the mobile charges. This representation is valid as long as $\mathbf{K}\mathbf{u}(\mathbf{K}) \ll 1$, where $\mathbf{u}(\mathbf{K})$ is the Fourier transform of the displacement field $\mathbf{u}$ and $\mathbf{K}$ is the wavevector. The disturbance of the charge density is given by $\rho = endiv\mathbf{u}$ and the current density is given by $\mathbf{j} = -en\dot{\mathbf{u}}$. The displacement field is subjected to the equation of motion

$$M\ddot{\mathbf{u}} = -e\mathbf{E} - e\mathbf{E}_0 \ , \tag{1}$$

where $\mathbf{E}$ is the internal electric field and $\mathbf{E}_0$ is an external electric field. In the non-retarded limit (where the body size is much smaller than the relevant electromagnetic wavelengths) the field $\mathbf{E}$ is the Coulomb field, so equation (1) can be written as

$$M\ddot{\mathbf{u}} = ngrad\int d\mathbf{r}'U(|\mathbf{r}-\mathbf{r}'|)div\mathbf{u}(\mathbf{r}') - e\mathbf{E}_0 \ , \tag{2}$$

where $U(r) = e^2/r$ is the Coulomb energy. In the retarded regime the field $\mathbf{E}$ is given by $\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - grad\Phi$, where $\mathbf{A}$ and Φ are the vector and, respectively, scalar electromagnetic potentials. They are by the well-known Kirchhoff radiation formulae

$$\mathbf{A} = \frac{1}{c}\int d\mathbf{r}'\frac{\mathbf{j}\left(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|/c\right)}{|\mathbf{r}-\mathbf{r}'|} \ , \ \Phi = \int d\mathbf{r}'\frac{\rho\left(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|/c\right)}{|\mathbf{r}-\mathbf{r}'|} \ , \tag{3}$$

where c is the velocity of light.

Making use of a temporal Fourier transform, equation (1) becomes

$$\omega^2\mathbf{u} = \frac{e}{M}\left(\mathbf{E} + \mathbf{E}_0\right) \ . \tag{4}$$

It is easy to see, by making use of the Maxwell equation $div\mathbf{E} = 4\pi\rho$, that equation (4) gives the well-known dielectric function $\varepsilon = 1 - \omega_p^2/\omega^2$ for a bulk plasma, where $\omega_p = \sqrt{4\pi ne^2/M}$ is the plasma frequency. The internal (polarizing) field is given by $\mathbf{E} = 4\pi ne\mathbf{u}$ (equal to $-4\pi\mathbf{P}$, where $\mathbf{P}$ is the polarization). Similarly, the equation of motion (4) and the Maxwell equation given above lead to the well-known conductivity $\sigma = ine^2/M\omega$. In this treatment we leave aside the magnetization, relativistic effects and dissipation.

Our general approach is to solve the integral equation (2) in the non-retarded limit in order to get the plasmon modes. Similarly, we introduce the displacement field $\mathbf{u}$ in equation (3) (via charge and current density) and use equation of motion (1) for expressing the electric field in terms of the displacement field. This way, we get integral equations (in general coupled with respect to the components of the displacement field), which we solve in order to get the electric field inside the body (*i.e.*, the refracted field). The eigenmodes of these equations give the polaritons. Having known the displacement field $\mathbf{u}$, we use again eqation (3) in order to calculate the field outside the body (*i.e.*, reflected, transmitted or diffracted field). We note that the use of integral equations in treating the electromagnetic field interacting with matter was previously indicated in connection with the so-called Ewald-Oseen extinction theorem[47].

We can add to equation (1) terms like $-M\omega_0^2\mathbf{u} - M\gamma\dot{\mathbf{u}}$, where ω_0 is a characteristic frequency and γ is a dissipation parameter. This amounts to replacing ω^2 in equation (4) by $\omega^2 - \omega_0^2 + i\gamma\omega$. We can view ω_p, ω_0 and γ as free parameters, thus being able to simulate various models of matter. For $\omega_0 = \gamma = 0$ we get the well-known dielectric function of an ideal plasma; if $\omega_0 = 0$ we have the dielectric function of the optical properties of simple metals for $\omega \gg \gamma$ (Drude model), and the dielectric function corresponding to the static (or quasi-static) currents in metals for $\omega \ll \gamma$; for $\omega_0 \gg \omega_p$ we have a dielectric function of usual dielectrics with loss; and so on. The dielectric function obtained by such an equation of motion is well known in the elementary theory of dispersion[48], and it provides support for the theory of the "effective medium permittivity". Herein, we limit ourselves to the equation of motion (1) for an ideal plasma, and apply it to cylindrical geometries.

2 Two-dimensional plasma sheet

We consider an infinite two-dimensional plasma sheet in the (x, y)-plane ($z = 0$) with a surface charge concentration n_s. For reasons of dimensionality we introduce a small distance d and express the surface concentration as $n_s = nd$, where n is the bulk charge concentration. The displacement field $\mathbf{u}$ lies in the (x, y)-plane, and we denote the position vector by $\mathbf{r} = (x, y)$. We use Fourier transforms of the form

$$\mathbf{u}(\mathbf{r}, t) = \sum_{\mathbf{k}} \int d\omega \mathbf{u}(\mathbf{k}\omega) e^{i\mathbf{kr}} e^{-i\omega t} \tag{5}$$

and the well-known Fourier decomposition

$$\frac{1}{r} = \sum_{\mathbf{k}} \frac{2\pi}{k} e^{i\mathbf{kr}} \tag{6}$$

for the Coulomb potential. We use a similar Fourier transform $\mathbf{E}_0(\mathbf{k}\omega)$ for the external field $\mathbf{E}_0(\mathbf{r}, t)$. For simplicity we often leave aside the arguments $\mathbf{k}$, ω of the Fourier transforms.

The Fourier transform of equation (2) gives

$$M\omega^2\mathbf{u} = \frac{2\pi ne^2 d}{k}(\mathbf{ku})\mathbf{k} + e\mathbf{E}_0 \ , \tag{7}$$

whence

$$\mathbf{u} = \frac{\omega_p^2 ed}{kM\omega^2} \cdot \frac{\mathbf{kE}_0}{2\omega^2 - \omega_p^2 kd}\mathbf{k} + \frac{e}{M\omega^2}\mathbf{E}_0 \ , \tag{8}$$

where $\omega_p = \sqrt{4\pi ne^2/M}$ is the plasma frequency. We can see the well-known longitudinal ($\mathbf{u} \sim \mathbf{k}$) plasmon mode $\omega = \omega_p\sqrt{kd/2}$. A plasma slab of thickness d has two surface-plasmon branches given[34] by $\omega^2 = \omega_p^2(1 \pm e^{-kd})/2$; one of them reduces to $\omega_p\sqrt{kd/2}$ in the limit $kd \ll 1$. The other branch, which reduces to ω_p, has no correspondent for the plasma sheet.

In the retarded regime we consider an incident plane wave $\mathbf{E}_0 e^{i\mathbf{kr}} e^{i\kappa z} e^{-i\omega t}$, with frequency $\omega = cK$ and wavevector $\mathbf{K} = (\mathbf{k}, \kappa)$ (satisfying the transversality condition $\mathbf{KE}_0 = 0$). For the "retarded" Coulomb potential we use the well-known decomposition[49]

$$\frac{e^{i\frac{\omega}{c}\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}} = \sum_{\mathbf{k}} \frac{2\pi i}{\kappa} e^{i\mathbf{kr}} e^{i\kappa|z|} \ , \tag{9}$$

where $\kappa = \sqrt{\omega^2/c^2 - k^2}$. From equation of motion (1) and the retarded potentials given by equation (3) we get

$$M\omega^2\mathbf{u} = \frac{2\pi i n e^2 d}{\kappa}(\mathbf{ku})\mathbf{k} - \frac{2\pi i n e^2 d\omega^2}{c^2\kappa}\mathbf{u} + e\mathbf{E}_0 \ . \tag{10}$$

It is convenient to introduce longitudinal and transverse components through $u_1 = \mathbf{ku}/k$ and, respectively, $u_2 = \mathbf{k}_\perp \mathbf{u}/k$, where $\mathbf{k}_\perp$ is a vector perpendicular to the vector $\mathbf{k}$, $\mathbf{kk}_\perp = 0$, and of the same magnitude k. Similarly, we use notations $E_{01} = \mathbf{kE}_0/k$ and $E_{02} = \mathbf{k}_\perp \mathbf{E}_0/k$. Then, equation (10) gives

$$\begin{aligned} u_1 &= \tfrac{2e}{M} \cdot \tfrac{E_{01}}{2\omega^2 + i\omega_p^2\kappa d} \ , \\ u_2 &= \tfrac{2e\kappa}{M} \cdot \tfrac{E_{02}}{2\kappa\omega^2 + i\omega_p^2 K^2 d} \ . \end{aligned} \tag{11}$$

We use these components of the displacement field in equation (3) for calculating the reflected field $\mathbf{E}_r$ (region $z < 0$) and the transmitted field $\mathbf{E}_t$ (region $z > 0$). The results of this calculation are

$$\begin{aligned} E_{r1,3} &= -\tfrac{i\omega_p^2\kappa d}{2\omega^2 + i\omega_p^2\kappa d} E_{01,3} e^{-i\kappa z} \ , \\ E_{r2} &= -\tfrac{i\omega_p^2 K^2 d}{2\kappa\omega^2 + i\omega_p^2 K^2 d} E_{02} e^{-i\kappa z} \end{aligned} \tag{12}$$

and

$$\begin{aligned} E_{t1,3} &= \tfrac{2\omega^2}{2\omega^2 + i\omega_p^2\kappa d} E_{01,3} e^{i\kappa z} \ , \\ E_{t2} &= \tfrac{2\omega^2\kappa}{2\kappa\omega^2 + i\omega_p^2 K^2 d} E_{02} e^{i\kappa z} \ , \end{aligned} \tag{13}$$

where we have preserved explicitly only the $e^{\pm i\kappa z}$-dependence. The component denoted by label 3 is directed along the z-axis. These fields cannot be obtained as the limiting case $d \to 0$ of a plasma slab of thickness d[34].

The (total) field inside the sheet is given by equation (4) as $\mathbf{E}_{tot} = M\omega^2\mathbf{u}/e$. It is worth noting that this field is equal to the transmitted field $\mathbf{E}_t$ (components 1 and 2). We can check the continuity of the fields at $z = 0$ in the form $\mathbf{E}_0 + \mathbf{E}_r = \mathbf{E}_t$. All the fields given above are propagating fields (κ is real), there is no damped regime, or a total reflection, etc, as expected. The denominators in equations (11)-(13) do not vanish, so we have no polaritonic modes. The reflection coefficient $R = |\mathbf{E}_r|^2/E_0^2$ decreases monotonically with increasing ω ($R(\omega = 0) = 1$) and the transmission coefficient $T = |\mathbf{E}_t|^2/E_0^2$ has a monotonical increase to unity for $\omega \to \infty$.

3 Circular aperture and disk

We consider a two-dimensional plasma sheet with a circular aperture of radius a. In cilindrical coordinates the displacement field can be written as

$$\mathbf{u} \rightarrow \mathbf{u}\theta(\rho - a) \ , \tag{14}$$

where $\theta(z) = 1$ for $z > 0$ and $\theta(z) = 0$ for $z < 0$ is the step function. We use the Fourier transform

$$\mathbf{u}(\rho, \varphi) = \sum_m e^{im\varphi}\mathbf{u}(\rho, m) \ , \tag{15}$$

where the summation extends over all integers m, as well as the decomposition[50]

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_m e^{im(\varphi - \varphi')} \int_0^\infty dk J_m(k\rho) J_m(k\rho') \tag{16}$$

for the Coulomb potential. We compute the divergence and the gradient occuring in equation (2) and notice that the resulting coupled integral equations for the components u_ρ and u_φ imply $imu_\rho = \frac{\partial}{\partial\rho}(\rho u_\varphi)$. This relation arises from $(curl\mathbf{u})_\rho = 0$. We use this relation to get only one integral equation for the component $u_\varphi(\rho, m)$. We leave aside the label φ and argument m and denote this component simply by $u(\rho)$. In the equation obeyed by function $u(\rho)$ we perform a few convenient integrations by parts and use the Bessel equation

$$z^2 \frac{d^2 J_m}{dz^2} + z\frac{\partial J_m}{\partial z} + (z^2 - m^2) J_m = 0 \tag{17}$$

to get

$$\begin{gathered} \tfrac{2\omega^2}{\omega_p^2 d}\rho u(\rho) = \textstyle\int_0^\infty k^2 dk \int_a^\infty \rho'^2 d\rho' J_m(k\rho) J_m(k\rho') u(\rho') - \\ -a^2 u(a) \textstyle\int_0^\infty kdk J_m(k\rho) J_m'(ka) - \tfrac{im}{2\pi ned}\Phi \ , \end{gathered} \tag{18}$$

where $\Phi = \Phi(\rho, m)$ is the corresponding Bessel-Fourier component of the external potential ($\mathbf{E}_0 = -grad\Phi$). Here it is convenient to introduce $\rho u(\rho) = v(\rho)$ and get

$$\begin{gathered} \tfrac{2\omega^2}{\omega_p^2 d} v(\rho) = \textstyle\int_0^\infty k^2 dk \int_a^\infty \rho' d\rho' J_m(k\rho) J_m(k\rho') v(\rho') - \\ -av(a) \textstyle\int_0^\infty kdk J_m(k\rho) J_m'(ka) - \tfrac{im}{2\pi ned}\Phi \ . \end{gathered} \tag{19}$$

We show below that this equation has no solution, because of the ultraviolet divergencies occurring for $k \rightarrow \infty$. Indeed, we use a Bessel-Fourier decomposition of the form

$$v(\rho) = \int_0^\infty kdk J_m(k\rho) v(k) \ , \ v(k) = \int_0^\infty \rho d\rho J_m(k\rho) v(\rho) \tag{20}$$

(and a similar one for the external potential). Equation (19) becomes

$$\begin{gathered} \left(-\tfrac{2\omega^2}{\omega_p^2 d} + k\right) v(k) = \\ \textstyle\int_0^\infty \lambda d\lambda v(\lambda) \left[ka^2 \int_0^1 xdx J_m(kax) J_m(\lambda ax) + aJ_m'(ka) J_m(\lambda a)\right] + \tfrac{im}{2\pi ned}\Phi(k) \ , \end{gathered} \tag{21}$$

where we have used the change of variable $\rho' = ax$. Here we use the inversion formulae[51]

$$v(k) = \int_0^1 xdx J_m(kax) v(x) \ , \ v(x) = a^2 \int_0^\infty kdk J_m(kax) v(k) \tag{22}$$

and get

$$-\frac{2\omega^2}{\omega_p^2 d}\int_0^1 xdxJ_m(kax)v(x)=\frac{1}{2a}J_m^{'}(ka)v(1)+\frac{im}{2\pi ned}\Phi(k)\ ; \tag{23}$$

we multiply both sides of this equation by $kJ_m(kax)$ and integrate with respect to k; equation (23) leads to

$$-\frac{2\omega^2}{\omega_p^2 da^2}v(x)=\frac{1}{2a}v(1)\int_0^\infty kdkJ_m(kax)J_m^{'}(ka)+\frac{im}{2\pi ned}\int_0^\infty kdkJ_m(kax)\Phi(k)\ . \tag{24}$$

Making use of a well-known recurrence formula for the Bessel functions[52], the integral involving $J_m^{'}$ in equation (24) can be reduced to two integrals involving products $J_m(kax)J_{m\pm 1}(ka)$. Such integrals are computed in Refs. [53, 54], with the aid of hypergeometric functions of argument x^2. For solving equation (24) we are interested in the limit $x\rightarrow 1$. We get

$$\lim_{x\rightarrow 1}\int_0^\infty kdkJ_m(kax)J_m^{'}(ka)=-\frac{2}{\pi a^2}\lim_{x\rightarrow 1}\frac{x}{1-x^2}\ \ , \tag{25}$$

which is divergent. We conclude that equation (24) has no solution.

There exists another decomposition of the Coulomb potential ([50], p. 126),

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{2}{\pi}\sum_m\int_0^\infty dke^{im(\varphi-\varphi')}I_m(k\rho_<)K_m(k\rho_>)\ \ , \tag{26}$$

where I_m and K_m are the modified Bessel functions of order m, and $\rho_<=\min(\rho,\rho')$, $\rho_>=\max(\rho,\rho')$. Doing the same calculations as above, by using equation (26), we reach the same conclusion: there exist ultraviolet divergencies, which prevent any finite solution. A similar situation holds for a circular disk.

For diffraction by the aperture we need a convenient decomposition of the "retarded" Coulomb potential. Making use of equation (9) we get easily

$$\frac{e^{i\frac{\omega}{c}\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}}=i\int_0^\infty kdkJ_0(k\rho)\frac{e^{i\kappa|z|}}{\kappa}\ \ , \tag{27}$$

where $\kappa=\sqrt{\omega^2/c^2-k^2}$. Using the well-known addition formula

$$J_0(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')})=\sum_{m=-\infty}^{+\infty}J_m(k\rho)J_m(k\rho')e^{im(\varphi-\varphi')}\ \ , \tag{28}$$

we get

$$\frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}=i\sum_{m=-\infty}^{+\infty}e^{im(\varphi-\varphi')}\int_0^\infty kdkJ_m(k\rho)J_m(k\rho')\frac{e^{i\kappa|z|}}{\kappa}\ . \tag{29}$$

We are not aware of a similar decomposition in terms of modified Bessel functions. We use the decomposition given by equation (29) for computing the electromagnetic potentials $\mathbf{A}$ and Φ given by equation (3) and the electric field $\mathbf{E}$ which appears into equation (1). Introducing the notation

$$f(\rho,\rho')=\int_0^\infty kdkJ_m(k\rho)J_m(k\rho')\frac{1}{\kappa}\ \ , \tag{30}$$

the equation of motion (1) can be written as

$$\begin{gathered}\frac{2i\omega^2}{\omega_p^2 d}u_\rho=\int_a^\infty\rho'd\rho'u_\rho\left(-\frac{\partial^2 f}{\partial\rho\partial\rho'}+\frac{\omega^2}{c^2}f\right)+im\int_a^\infty d\rho'u_\varphi\frac{\partial f}{\partial\rho}+\frac{i}{2\pi ned}E_{0\rho}\ ,\\ \frac{2\omega^2}{m\omega_p^2 d}\rho u_\varphi=-\int_a^\infty\rho'd\rho'u_\rho\frac{\partial f}{\partial\rho'}+\int_a^\infty d\rho'u_\varphi\left(im-\frac{i\omega^2}{mc^2}\rho\rho'\right)f+\frac{\rho}{2\pi mned}E_{0\varphi}\ ,\end{gathered} \tag{31}$$

where we have left aside the arguments of the functions in the integrals. We were not able to solve these equations. The same situations holds for another decomposition[55]

$$\frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}=\frac{i}{2}\sum_{m=-\infty}^{+\infty}e^{im(\varphi'-\varphi)}e^{i\kappa(z'-z)}\int_0^{\infty}\kappa d\kappa J_m(k\rho)H_m(k\rho')\ ,\ \rho'>\rho\ . \tag{32}$$

4 Infinite cylindrical hole and rod

We consider first an infinite circular cylindrical hole of radius a extending from $z=-\infty$ to $z=+\infty$. We use cylindrical coordinates $\rho\,,\varphi\,,z$. We write the displacement field $\mathbf{u}$ as in equation (14), and use the same Fourier transform given by equation (15). We proceed in the same way as we did for the circular aperture above, compute the divergence of the displacement field (in cylindrical coordinates) and the gradient occurring in equation (2), and note the occurence of specific contributions arising from the boundary $\rho=a$. First, we use the decomposition[50]

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_m e^{im(\varphi-\varphi')}\int_0^{\infty}dke^{-k(z_>-z_<)}J_m(k\rho)J_m(k\rho')\ , \tag{33}$$

where J_m is the Bessel function of the first kind and order m, $z_<=\min(z,z')$, $z_>=\max(z,z')$. Equation (2) gives three coupled integral equations for the components $u_\rho\,,u_\varphi\,,u_z$, and we can see easily that they imply the relations

$$imu_\rho=u_\varphi+\rho\frac{\partial u_\varphi}{\partial\rho}\ ,\ imu_z=\rho\frac{\partial u_\varphi}{\partial z}\ ; \tag{34}$$

these relations arises from $(curl\mathbf{u})_\rho=0$ and $(curl\mathbf{u})_z=0$. We use these relations to get only one integral equation for $u_\varphi(\rho,m,z)$; we denote this function simply by $u(\rho,z)$. After a few convenient integrations by parts and using the Bessel equation (17), we get the following integral equation for function $u(\rho,z)$:

$$\begin{aligned}\tfrac{2}{\omega_p^2}(\omega_p^2-\omega^2)\rho u(\rho,z)=a^2\textstyle\int_0^\infty kdkJ_m(k\rho)J_m'(ka)\int_{-\infty}^{+\infty}dz'u(a,z')e^{-k|z-z'|}-\\ -\tfrac{\rho}{2\pi ne}E_{0\varphi}(\rho,z)\ .\end{aligned} \tag{35}$$

Here, it is convenient to make a Fourier transform with respect to coordinate z. We get easily

$$(\omega^2-\omega_p^2)u(\rho,\kappa)=\tfrac{e}{M}E_{0\varphi}(\rho,\kappa)-\tfrac{e}{M}E_{0\varphi}(a,\kappa)\tfrac{\omega_p^2aI(\rho)}{\omega^2-\omega_p^2+\omega_p^2aI(a)}\ , \tag{36}$$

where

$$I(\rho)=\int_0^\infty dk\frac{k^2}{k^2+\kappa^2}J_m(k\rho)J_m'(ka)\ . \tag{37}$$

Making use of the well-known recurrence relations for the Bessel functions,[52] this integral can be brought to a known integral[56]. We get

$$I(\rho)=\frac{\kappa}{2}K_m(\kappa\rho)\left[I_{m-1}(\kappa a)+I_{m+1}(\kappa a)\right]=\kappa K_m(\kappa\rho)I_m'(\kappa a)\ , \tag{38}$$

where I_m and K_m are the modified Bessel functions of degree m.

It folows that the bulk plasmons are given by $\omega^2=\omega_p^2$ and the surface plasmons are given by

$$\omega^2=\omega_p^2\left[1-\kappa aK_m(\kappa a)I_m'(\kappa a)\right]\ . \tag{39}$$

This relation is symmetric under the transformation $m \to -m$. In the limit $\kappa a \ll 1$ we get

$$\omega^2 \simeq \begin{cases} \omega_p^2 \left[1 + \frac{(\kappa a)^2}{2} \ln \frac{\kappa a}{2}\right] \ , \ m = 0 \\ \frac{1}{2}\omega_p^2 \left[1 - \frac{(\kappa a)^2}{2} \ln \frac{\kappa a}{2}\right] \ , \ m = 1 \\ \frac{1}{2}\omega_p^2 \left[1 + \frac{(\kappa a)^2}{2m(m^2-1)}\right] \ , \ m > 1 \end{cases} . \tag{40}$$

Having known the displacement field $\mathbf{u}$ given by equations (34) and (36), and using equation (14), we can compute the polarization potential

$$\Phi_p(\mathbf{r}) = ne \int d\mathbf{r}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} div\mathbf{u} \tag{41}$$

and the total potential $\Phi = \Phi_p + \Phi_0$, in terms of the external potential Φ_0 (via the external field $\mathbf{E}_0$), getting thus the dielectric response. The result of this calculation is

$$\begin{gathered} \Phi_i(\rho, \varphi, z) = \Phi_0(\rho, \varphi, z) - \\ -a\omega_p^2 \sum_m e^{im\varphi} \int d\kappa e^{i\kappa z} I_m(\kappa\rho) \frac{\kappa K_m'(\kappa a)\Phi_0(a, m, \kappa)}{\omega^2 - \omega_p^2 + \omega_p^2 \kappa a K_m(\kappa a) I_m'(\kappa a)} \end{gathered} \tag{42}$$

for $\rho < a$ (inside the hole) and

$$\begin{gathered} \Phi_e(\rho, \varphi, z) = \frac{\omega^2}{\omega^2 - \omega_p^2} [\Phi_0(\rho, \varphi, z) - \\ -a\omega_p^2 \sum_m e^{im\varphi} \int d\kappa e^{i\kappa z} K_m(\kappa\rho) \frac{\kappa I_m'(\kappa a)\Phi_0(a, m, \kappa)}{\omega^2 - \omega_p^2 + \omega_p^2 \kappa a K_m(\kappa a) I_m'(\kappa a)}] \end{gathered} \tag{43}$$

for $\rho > a$ (outside the hole). We can check that this potential and its derivatives $\partial\Phi/\partial z$ and $\partial\Phi/\partial\rho$ are continuous at the hole surface $\rho = a$, which means the continuity of the tangential components of the electric field, but the normal component $\partial\Phi/\partial\rho$ is discontinuous, as expected. For an external field along the axis of the hole, as well as for an external field along the radius, we can check the continuity of the normal component of the electric displacement $E_i = \varepsilon(\omega) E_e |_{\rho=a}$ where $\varepsilon = 1 - \omega_p^2/\omega^2$ is the dielectric function of a bulk ideal plasma. In this respect, we may say that the so-called theory of "effective medium permittivity" holds. In addition, we may note the occurrence of specific surface contributions to the dielectric response (the integrals in equations (42) and (43)), which precludes in fact a proper definition for a conventional dielectric function.

Similar calculations can be performed for an infinite plasma rod. In this case, we get the surface plasmons

$$\omega^2 = \omega_p^2 \left[1 + \kappa a I_m(\kappa a) K_m'(\kappa a)\right] \ , \tag{44}$$

which in the limit $\kappa a \ll 1$ reads

$$\omega^2 \simeq \begin{cases} -\frac{1}{2}\omega_p^2 (\kappa a)^2 \ln \frac{\kappa a}{2} \ , \ m = 0 \quad {}^2 \\ \frac{1}{2}\omega_p^2 \left[1 + \frac{(\kappa a)^2}{2} \ln \frac{\kappa a}{2}\right] \ , \ m = 1 \\ \frac{1}{2}\omega_p^2 \left[1 - \frac{(\kappa a)^2}{2m(m^2-1)}\right] \ , \ m > 1 \end{cases} . \tag{45}$$

The dielectric response reads

$$\Phi_i(\rho,\varphi,z)=\tfrac{\omega^2}{\omega^2-\omega_p^2}[\Phi_0(\rho,\varphi,z)+$$
$$+a\omega_p^2\sum_m e^{im\varphi}\int d\kappa e^{i\kappa z}I_m(\kappa\rho)\tfrac{\kappa K_m'(\kappa a)\Phi_0(a,m,\kappa)}{\omega^2-\omega_p^2-\omega_p^2\kappa a I_m(\kappa a)K_m'(\kappa a)}] \tag{46}$$

for $\rho<a$ and

$$\Phi_e(\rho,\varphi,z)=\Phi_0(\rho,\varphi,z)+$$
$$+a\omega_p^2\sum_m e^{im\varphi}\int d\kappa e^{i\kappa z}K_m(\kappa\rho)\tfrac{\kappa I_m'(\kappa a)\Phi_0(a,m,\kappa)}{\omega^2-\omega_p^2-\omega_p^2\kappa a K_m(\kappa a)I_m'(\kappa a)} \tag{47}$$

for $\rho>a$.

These plasmon modes given above can be called "cylindrical" plasmons. It is worth noting that we cannot get the bulk plasma frequency ω_p in the limit $a\to 0$, because of the surface charge which is not vanishing in this limit. The results given in equations (39) and (44) coincide with those given in Refs. [42, 44].

We can use the other decomposition ([50], p. 126)

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{2}{\pi}\sum_m\int_0^\infty dk e^{im(\varphi-\varphi')}\cos k(z-z')I_m(k\rho_<)K_m(k\rho_>) \tag{48}$$

for the Coulomb potential and get the same results as those given by equations (39) and (44).

The diffraction of an electromagnetic field by an infinite cylindrical hole or rod can be treated in the same manner as for a circular aperture (or disk), following the general procedure described in **Introduction**. Unfortunately, we were not able to solve the resulting integral equations.

5 Finite cylindrical hole

We consider a circular cylindrical hole of radius a bored into a plasma slab of thickness d. The slab extends over the region $0<z<d$. We use cylindrical coordinates $\rho\,,\varphi\,,z$. The displacement field $\mathbf{u}$ can be written as

$$\mathbf{u}\to\mathbf{u}\,[\theta(z)-\theta(z-d)]\,\theta(\rho-a) \tag{49}$$

and use the same Fourier tranform given by equation (15). We compute the divergence of the displacement field $\mathbf{u}$ and the gradient occurring in equation (2), and note again the occurence of specific contributions arising from the boundaries $z=0$, d and $\rho=a$. Now, we use first the decomposition given by equation (48) ([50], p. 126)

Equation (2) gives three coupled integral equations for the components $u_\rho\,,u_\varphi\,,u_z$, and we can see easily that they imply the relations

$$imu_\rho=u_\varphi+\rho\frac{\partial u_\varphi}{\partial\rho}\ ,\ imu_z=\rho\frac{\partial u_\varphi}{\partial z}\ ; \tag{50}$$

these relations arises from $(curl\mathbf{u})_\rho=0$ and $(curl\mathbf{u})_z=0$. We use these relations to get only one integral equation for $u_\varphi(\rho,m,z)$; we denote this function simply by $u(\rho,z)$. After a few convenient integrations by parts and using the equation

$$z^2\frac{d^2w}{dz^2}+z\frac{\partial w}{\partial z}-(z^2+\nu^2)w=0 \tag{51}$$

for the modified Bessel functions w_ν we get the following integral equation for the function $u(\rho, z)$:

$$\begin{gathered}\frac{2\pi}{\omega_p^2}(\omega_p^2-\omega^2)\rho u(\rho,z)=a^2\int_0^\infty kdk\int_0^d dz'u(a,z')\cos k(z-z')K_m(k\rho)I_m'(ka)+\\+\int_0^\infty kdk\int_a^\rho d\rho'\rho'^2\left[u(\rho',0)\sin kz-u(\rho',d)\sin k(z-d)\right]K_m(k\rho)I_m(k\rho')+\\+\int_0^\infty kdk\int_\rho^\infty d\rho'\rho'^2\left[u(\rho',0)\sin kz-u(\rho',d)\sin k(z-d)\right]I_m(k\rho)K_m(k\rho')-\\-\frac{1}{2ne}E_{0\varphi}(\rho,z)\ ,\end{gathered} \tag{52}$$

where $I_m'(ka)$ is the derivative of the modified Bessel function with respect to its argument. This equation relates the function $u(\rho, z)$ to its values $u(a, z)$, $u(\rho, 0)$, $u(\rho, d)$ on the boundaries. We should particularize equation (52) to $\rho = a$, $z = 0$ and $z = d$. This way, we obtain a set of three coupled integral eqautions, which, unfortunately, we were not able to solve. A similar situation occurs for the decomposition given by equation (33), as well as for the diffracted field.

6 Discussion and conclusions

We used the equation of motion for matter polarization (elementary theory of classical dispersion) and Kirchhoff radiation potentials to investigate plasmons, polaritons and diffracted electromagnetic field in matter, especially in an ideal model of electronic jellium-like plasma. Typically, this method leads to coupled integral equations for the components of the displacement (polarization) field. This integral-equation method was sought for long in studying the interaction of the electromagnetic field with matter[47]. It is particularly interesting for structures with restricted geometries and for giving support to the theory of the "effective medium permittivity".

We have applied herein this method to structures with cylindrical geometries. First, we have computed the plasmon modes and the diffracted field for a two-dimensional plasma sheet. Next, we have shown that unphysical ultraviolet divergencies occur for plasmons in an infinite plasma sheet with a circular aperture (or a circular disk). We were not able to decouple the integral equations for diffraction by an aperture (or a disk). Further on, we have computed the "cylindrical" plasmons occurring in an infinite cylindrical hole and an infinite cylindrical rod. Unfortunately, we were not able to solve the coupled integral equations for plasmons or for the diffraction problem for a finite cylindrical hole, or the difracted field by an infinite cylindrical hole (or rod).

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