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van derWaals-London electromagnetic forces involving spherical bodies

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Abstract

Electromagnetic forces are computed in the non-retarded regime (van der Wals-London forces) for any pair of the following bodies: one half-space (a semi-infinite solid with a plane surface), spheres, spherical shells and point-like particles. The force acting beween a half-space and any other body considered here goes like d^{-4} (for two half-spaces it goes like $1/d^3$), where d is the separation distance between the bodies. The force acting between two spheres goes like $1/d^5$. These forces are attractive. The spherical shells and the point-like bodies behave very much alike the spheres, except for the lack of (internal) polarization. This latter (rather unrealistic) feature introduces some particularities. Whenever a "conducting" particle of this type appears (like "conducting" spherical shells or point-like bodies), the force is repulsive. For instance, a "conducting" spherical shell, or a "conducting" point-like body, gives always a repulsive force. Two "conducting" spherical shells, or the pair of "conducting" spherical shell-point-like body, or two "conducting" point-like bodies interact with a repulsive force which goes like $d^{-7/2}$. The calculations are performed within the Lorentz-Drude (plasma) model of polarizable matter, by using the dipole approximation for the spherical particles. The coupled equations of motion of the polarization are obtained for each pair of bodies and solved for the electromagnetic eigenfrequencies. The force is derived from the variation of the zero-point energy of the electromagnetic field with the separation distance.

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Introduction

As it is well-known,[1, 2] the van der Waals-London and Casimir forces have been derived originally by quantum-mechanical calculations in the non-retarded (small distances) and, respectively, retarded (longer distances) regime (following the original quantum-mechanical derivation of the van der Waals-London forces[3]-[5]). We focus in this paper on the non-retarded regime (van der Waals-London force), which is more accessible experimentally. The van der Waals-London force acting between a quantum particle (neutral atom) and a half-space (a semi-infinite solid with a plane surface) goes like $1/d^4$, where d is the separation distance between the two bodies. For a pair of neutral atoms the force goes like $1/d^7$. Subsequently, such results have been re-derived by using the quantum-statistical theory of the electromagnetic fluctuations[6]-[8] and the source theory.[9, 10] In particular, the force acting between two half-spaces have been obtained, which goes like $1/d^3$. In general, if the force acting between two particles goes like d^{-n}, then the force between a particle and a half-space goes like d^{-n+3} (and the force acting between two half-spaces goes like d^{-n+4}). As it is well known, the origin of these forces resides in the zero-point energy of the electromagnetic field (vacuum fluctuations).

The need for similar electromagnetic forces acting between macroscopic bodies has been pointed out long ago.[11]-[15] The macroscopic bodies bring their own characteristics with respect to the electrical polarization (like plasmons, polaritons, surface effects, etc), in comparison with individual quantum particles. On the other hand, the experimental interest lies mainly in macroscopic bodies, especially of finite size (like spheres).[16]-[35] A classical counterpart of the quantum van der Waals-London and Casimir forces is therefore expected.

A method for computing the electromagnetic forces acting between macroscopic, polarizable bodies has been put forward recently.[36, 37] It is based on representing the polarization by a slight displacement field $\mathbf{u}(\mathbf{R}, t)$ of the mobile charges (function of position $\mathbf{R}$ and the time t), subjected to the classical (Newton) equation of motion, within the Lorentz-Drude (plasma) model of matter polarization. The electromagnetic field is computed by using this polarization, and the equation of motion is solved for the displacement field $\mathbf{u}$. For interacting bodies we get coupled equations of motion, which are solved for the eigenfrequencies. The energy is computed as the correction to the zero-point energy of the electromagnetic field, as brought about by the interaction, and the force is thereby derived. Making use of this method, the Casimir force acting between two half-spaces has been derived ($\sim 1/d^4$), as well as the van der Waals-London force for two half-spaces ($\sim 1/d^3$)[36] and the van der Waals force acting between a point-like body and a half-space ($\sim 1/d^4$).[37] It was shown in Refs. [36, 37] that the Casimir force is governed by the surface plasmon-polariton modes, while the van der Waals–London force arises from surface plasmons.

We present here a series of computations for the van der Waals-London force involving spherical bodies (like spheres, spherical shells and point-like bodies) and a half-space. The van der Waals-London force for a half-space coupled to any body of the type considered here goes like $1/d^4$. The spherical shells and the point-like bodies behave very much alike the spheres, except for the lack of (internal) polarization. This rather unrealistic feature introduces some particularities. For instance, whenever such a "conducting" body appears, the force is repulsive. An interesting case occurs for a pair of "conducting" spherical shells, or the pair of "conducting" spherical shell-point-like body, or two "conducting" point-like bodies, where the force is repulsive and goes like $1/d^{7/2}$.

Lorentz-Drude model

The well-known Lorentz-Drude model[38]-[43] of (homogeneous) polarizable matter consists of identical charges q, with mass m and density n, moving in a rigid neutralizing background. A slight displacement field $\mathbf{u}(\mathbf{R}, t)$ is subjected to the equation of motion

$$m\ddot{\mathbf{u}} = q(\mathbf{E} + \mathbf{E}_0) - m\omega_c^2\mathbf{u} - m\gamma\dot{\mathbf{u}} \ , \tag{1}$$

where $\mathbf{E}$ is the electric field produced by the polarization charges and currents, $\mathbf{E}_0$ is an external electric field, ω_c is a characteristic frequency of the body and γ is a damping factor. For dielectrics $\omega_c \neq 0$, for conductors $\omega_c = 0$. Since the polarization is given by $\mathbf{P} = nq\mathbf{u}$, it is easy to see that equation (1) leads to the well-known electric susceptibility

$$\chi = -\frac{\omega_p^2}{4\pi} \cdot \frac{1}{\omega^2 - \omega_c^2 + i\omega\gamma} \tag{2}$$

and dielectric function $\varepsilon = 1 + 4\pi\chi$, where $\omega_p = \sqrt{4\pi nq^2/m}$ is the plasma frequency. We note the absence of the Lorenz force in equation (1), whose contribution is quadratic in the displacement field $\mathbf{u}$ and, consequently, it may be neglected. However, we can include an external magnetic field in equation of motion (1), if necessary.

The displacement field $\mathbf{u}$ produces polarization charge and current densities given by

$$\rho = -div\mathbf{P} = -nqdiv\mathbf{u} \ , \ \mathbf{j} = \frac{\partial \mathbf{P}}{\partial t} = nq\dot{\mathbf{u}} \ , \tag{3}$$

which can be used to compute the electromagnetic potentials

$$\begin{aligned} \Phi(\mathbf{R}, t) &= \int d\mathbf{R}' \frac{\rho(\mathbf{R}', t-|\mathbf{R}-\mathbf{R}'|/c)}{|\mathbf{R}-\mathbf{R}'|} \ , \\ \mathbf{A}(\mathbf{R}, t) &= \frac{1}{c} \int d\mathbf{R}' \frac{\mathbf{j}(\mathbf{R}', t-|\mathbf{R}-\mathbf{R}'|/c)}{|\mathbf{R}-\mathbf{R}'|} \end{aligned} \tag{4}$$

(subjected to the Lorenz gauge $div\mathbf{A} + (1/c)\partial\Phi/\partial t = 0$). These potentials give rise to the electric field $\mathbf{E}$ in equation (1), whence we can get the displacement $\mathbf{u}$. This way, we can compute the coupled polarization modes for interacting bodies, the external fields in equation (1) being the mutual fields by which the bodies act one upon another. The eigenfrequncies of these coupled equations of motion are the relevant frequencies for the zero-point energy of the electromagnetic field.

We focus here on the non-retarded regime (van der Waals-London forces), where the fields vary slowly over the size of the bodies, and the vector potential $\mathbf{A}$ (as well as the current density $\mathbf{j}$) may be neglected ($\omega l/c \ll 1$, where ω is the frequency of the fields and l is a scale of the bodies size). As it is well known, this approximation is also called the quasi-static approximation.

Half-space

For a half-space extending over the region $z > d$ we take the polarization as

$$\mathbf{P} = nq(\mathbf{u}, \, u_z)\theta(z-d) \ , \tag{5}$$

where $\theta(z) = 0$ for $z < 0$ and $\theta(z) = 1$ for $z > 0$ is the step function, and get the polarization charge density

$$\rho = -nq(div\mathbf{u} + \tfrac{\partial u_z}{\partial z})\theta(z-d) - nqu_z(d)\delta(z-d) \ . \tag{6}$$

We use the Fourier decomposition of the type

$$\mathbf{u}(\mathbf{r}, z; t) = \frac{1}{(2\pi)^3} \int d\mathbf{k} d\omega \mathbf{u}(k, z; \omega) e^{-i\omega t + i\mathbf{k}\mathbf{r}} \ , \tag{7}$$

and may omit ocassionally the arguments $\mathbf{k}$, ω, writing simply $\mathbf{u}(z)$, or even $\mathbf{u}$. As a rule, we omit everywhere the argument ω. Likewise, we leave aside the factor nq, but restore it in the final formulae. The well-known decomposition[44]

$$\frac{1}{|\mathbf{R}-\mathbf{R}'|} = \frac{1}{2\pi} \int d\mathbf{k} \frac{1}{k} e^{ik(\mathbf{r}-\mathbf{r}')} e^{-k|z-z'|} \ , \tag{8}$$

is used for the Coulomb potential. The calculations are straightforward and we get the Fourier tranforms of the scalar potential

$$\Phi(\mathbf{k}, z) = -\frac{2\pi i}{k} \int_d^\infty dz' \mathbf{k}\mathbf{u} e^{-k|z-z'|} - \frac{2\pi}{k} \frac{\partial}{\partial z} \int_d^\infty dz' u_z e^{-k|z-z'|} \ . \tag{9}$$

In order to compute the electric field it is convenient to refer the in-plane vectors (*i.e.*, vectors parallel with the surface of the half-space) to the vectors $\mathbf{k}$ and $\mathbf{k}_\perp$, where $\mathbf{k}_\perp$ is perpendicular to $\mathbf{k}$ and of the same magnitude as $\mathbf{k}$; for instance, we write

$$\mathbf{u} = u_1 \frac{\mathbf{k}}{k} + u_2 \frac{\mathbf{k}_\perp}{k} \tag{10}$$

and a similar representation for the electric field parallel with the surface of the half-space. It is worth paying attention to the correct derivative of the modulus function, according to the equation

$$\frac{\partial^2}{\partial z^2}e^{-|z-z'|} = k^2 e^{-k|z-z'|} - 2k\delta(z-z') \ . \tag{11}$$

We get the electric field

$$\begin{array}{c} E_1 = -ik\Phi = -2\pi k \int_d^\infty dz' u_1 e^{-k|z-z'|} + 2\pi i \frac{\partial}{\partial z}\int_d^\infty dz' u_z e^{-k|z-z'|} \ , \\ E_z = -\frac{\partial \Phi}{\partial z} = 2\pi i \frac{\partial}{\partial z}\int_d^\infty dz' u_1 e^{-k|z-z'|} + 2\pi k \int_d^\infty dz' u_z e^{-k|z-z'|} - 4\pi`u_z\theta(z-d) \end{array} \tag{12}$$

(and $E_2 = 0$). We can check easily the equalities

$$ikE_1 + \frac{\partial E_z}{\partial z} = -4\pi\left(iku_1 + \frac{\partial u_z}{\partial z}\right)\theta(z-d) - 4\pi u_z(d)\delta(z-d) \ , \tag{13}$$

which is the Gauss's law, and

$$\frac{\partial E_1}{\partial z} - ikE_z = 0 \ , \tag{14}$$

which reflects the fact that the field $\mathbf{E}$ derives from the gradient of the potential ($\mathbf{E} = -grad\Phi$). From equation (13), we can check the transversality condition $div\mathbf{E} = 0$ for the electric field outside the half-space ($z < d$).

Making use of these relations (equations (13) and (14)) in the equation of motion (1) (with $\gamma = 0$ and for $z > d$), and taking into account that $div\mathbf{E}_0 = 0$ and $\partial E_{01}/\partial z - ikE_{0z} = 0$ in this equation, we get

$$\frac{\partial^2 u_1}{\partial z^2} - k^2 u_1 = 0 \ , \tag{15}$$

whose solution is $u_1 = A_1 e^{-kz}$ (and $u_z = iA_1 e^{-kz}$), where A_1 is a constant. We use these solutions for u_1 and u_z in equations (12) in order to compute the field. We get

$$\begin{array}{c} E_1 = -2\pi A_1 e^{-kz} \ , \ z > d \ , \\ E_1 = -2\pi A_1 e^{k(z-2d)} \ , \ z < d \end{array} \tag{16}$$

(and $ikE_z = \partial E_1/\partial z$, $E_2 = 0$ everywhere). The equation of motion (1) can now be written as

$$(\omega^2 - \omega_c^2 - \frac{1}{2}\omega_p^2)A_1 e^{-kz} = -\frac{q}{m}E_{01} \ , \ (\omega^2 - \omega_c^2)u_2 = -\frac{q}{m}E_{02} \ , \tag{17}$$

where we can recognize the frequency $\omega_p/\sqrt{2}$ of the surface plasmons.[45]

The sphere

We consider a sphere of radius a, with charge density $n_0 q$, placed at the origin. We leave aside for the moment the factor $n_0 q$, which will be restored in the final formulae. For small radii a, and according to the symmetry of the problem, the displacement field of the sphere can be taken as

$$\mathbf{u} = (\beta x, \ \beta y, \ \alpha a)\theta(a-R) \ , \tag{18}$$

where α and β are constants. Indeed, the field generated by the half-space in the region $z < d$ is given by the (inverse) Fourier transform of the second row in equations (16). For the in-plane field we get

$$\mathbf{E}_\perp(\mathbf{r}, z) = -\frac{A_1}{2\pi}\int d\mathbf{k}\frac{\mathbf{k}}{k}e^{i\mathbf{kr}}e^{k(z-2d)} \ , \tag{19}$$

and a similar equation for the component $E_z(\mathbf{r}, z)$. The integral in equation (19) implies the Bessel function of the first order $J_1(kr)$, and the remaining integral is given in Ref. [44], p. 686 (6.611.1). We get the fields

$$E_x \simeq -i\frac{A_1}{8d^3}x \ , \ E_y \simeq -i\frac{A_1}{8d^3}y \ , \ E_z \simeq i\frac{A_1}{4d^2} \ , \tag{20}$$

generated by the half-space in the region of the sphere (for $a \ll d$). Therefore, the displacement field $\mathbf{u}$ chosen for the sphere in equation (18) is justified. The charge density corresponding to the displacement field given by equation (18) ($\rho = -div\mathbf{u}$) is

$$\begin{gathered}\rho = -div\mathbf{u} = \\ = -2\beta\theta(a-R) + \tfrac{2}{3}\beta a\delta(a-R) + \alpha a P_1(\cos\theta)\delta(a-R) - \tfrac{2}{3}\beta a P_2(\cos\theta)\delta(a-R) \ ,\end{gathered} \tag{21}$$

where $P_n(\cos\theta)$ are the Legendre polynomials and $\cos\theta = z/R$ ($= z/a$). Within the dipole approximation we may neglect the P_2-term in the above equation, which amounts to putting $\beta = 0$. Consequently, the displacement field reduces to $\mathbf{u} = \alpha a(0, 0, 1)\theta(a-R)$ and the charge density is $\rho = \alpha a P_1(\cos\theta)\delta(a-R)$. We compute the (non-retarded) scalar potential given by equations (4) for this charge density, by using the well-known decompositon of the Coulomb potential

$$\frac{1}{|\mathbf{R}-\mathbf{R}'|} = \sum_{n=0} \frac{R_<^n}{R_>^{n+1}} P_n(\cos\Theta) \tag{22}$$

and the addition formula (Ref. [44], p. 965, 8.814)

$$P_n(\cos\Theta) = P_n(\cos\theta)P_n(\cos\theta') + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\varphi-\varphi') \ , \tag{23}$$

where P_n^m are the associated Legendre functions, $R_< = \min(R, R')$, $R_> = \max(R, R')$, (θ, φ) define the direction of $\mathbf{R}$, (θ', φ') define the direction of $\mathbf{R}'$ and Θ ($\cos\Theta = \sin\theta\sin\theta'\cos(\varphi - \varphi') + \cos\theta\cos\theta'$) is the angle between $\mathbf{R}$ and $\mathbf{R}'$. We get

$$\begin{gathered}\Phi(\mathbf{R}) = \tfrac{4\pi}{3}\alpha a z \ , \ R < a \ , \\ \Phi(\mathbf{R}) = \tfrac{4\pi}{3}\alpha a^4 \tfrac{z}{R^3} \ , \ R > a \ .\end{gathered} \tag{24}$$

The polarization field inside the sphere is $E_z = -(4\pi/3)\alpha a$, and the equation of motion (1) becomes

$$(\omega^2 - \omega_{c0}^2 - \frac{1}{3}\omega_{p0}^2)\alpha a\theta(a-R) = -\frac{q}{m}E_{0z} \ , \tag{25}$$

where ω_{c0} denotes the characteristic frequency of the sphere, $\omega_{p0} = \sqrt{4\pi n_0 q^2/m}$ is the plasma frequency of the sphere and the external field E_{0z} is the field generated by the half-space inside the sphere. The Fourier transform of this field, as given by equations (16), is $E_{0z}(\mathbf{k}, z) = 2\pi i A_1 e^{k(z-2d)} \simeq 2\pi i A_1 e^{-2kd}$ ($z \ll d$). The Fourier transform of the function $\theta(a-R)$ entering equation (25) can be computed easily. For $ak \ll 1$ it is given by πa^2, so that equation (25) becomes

$$(\omega^2 - \omega_{c0}^2 - \frac{1}{3}\omega_{p0}^2)\alpha a^3 = -\frac{i}{2\pi}\omega_p^2 A_1 e^{-2kd} \ . \tag{26}$$

We recognize in equation (26) the lowest (dipole) frequency $\omega_{p0}/\sqrt{3}$ of the spherical plasmons.[46]

The field produced by the sphere in the region of the half-space (the external field for the half-space) can be obtained from equations (24) (for $R > a$). We notice that $z/R^3 = -\partial(1/R)/\partial z$, and the Fourier transform of the function $f(R) = 1/R$ is[44]

$$f(\mathbf{k}, z) = \int d\mathbf{r} \frac{1}{R} e^{-i\mathbf{kr}} = 2\pi \int_z dR J_0(k\sqrt{R^2 - z^2}) = 2\pi \frac{e^{-k|z|}}{k} \tag{27}$$

(where J_0 is the Bessel function of the zeroth order). Consequently, the field is given by

$$E_{01}(k,\, z) = -\frac{8\pi^2 i}{3}\alpha a^4 k e^{-kz} \;,\; E_{0z} = \frac{8\pi^2}{3}\alpha a^4 k e^{-kz} \tag{28}$$

(and $E_{02} = 0$) and equation (17) becomes

$$(\omega^2 - \omega_c^2 - \frac{1}{2}\omega_p^2)A_1 = \frac{2\pi i}{3}\omega_{p0}^2 \alpha a^4 k \tag{29}$$

(and $u_2 = 0$).

The two coupled equations (26) and (29) give the dispersion equation

$$(\omega^2 - \omega_c^2 - \frac{1}{2}\omega_p^2)(\omega^2 - \omega_{c0}^2 - \frac{1}{3}\omega_{p0}^2) = \frac{1}{3}\omega_p^2\omega_{p0}^2 ake^{-2kd} \;. \tag{30}$$

Force between a sphere and a half-space

Within the dipole aproximation the coupling strength ak is very weak ($ak \ll 1$). We may consider the *rhs* of equation (30) as a small perturbation. Introducing the notations

$$A^2 = \omega_c^2 + \frac{1}{2}\omega_p^2 \;,\; B_0^2 = \omega_{c0}^2 + \frac{1}{3}\omega_{p0}^2 \tag{31}$$

the solutions of equation (30) can be written as

$$\begin{aligned} \omega_1 &\simeq A + \tfrac{\omega_p^2\omega_{p0}^2}{6A(A^2-B_0^2)} ake^{-2kd} \;, \\ \omega_2 &\simeq B_0 - \tfrac{\omega_p^2\omega_{p0}^2}{6B(A^2-B_0^2)} ake^{-2kd} \end{aligned} \tag{32}$$

(we note that $A \neq B_0$). In the non-retarded limit we should impose in fact the conditions A, $B_0 \ll \omega_c$, ω_{c0}, ω_p, ω_{p0} (and Aa/c, $B_0a/c \ll 1$). However, bearing in mind that we are interested only in the change in energy brought about by the interaction, we may leave aside such restrictive conditions, and view, indeed, the *rhs* of equation (30) as a small perturbation. This is a special feature of the classical approach, in contrast with quantum-mechanical calculations, where we have direct access to the perturbation energy (see, for instance, Ref. [12]). Making use of equations (32), we compute the change ΔE brought about by the interaction in the zero-point energy of the electromagnetic field (per unit area). We get

$$\Delta E = -\frac{\hbar a}{96\pi} \cdot \frac{\omega_p^2\omega_{p0}^2}{AB_0(A+B_0)} \cdot \frac{1}{d^3} \tag{33}$$

and the force (leading contribution in powers of $1/d$)

$$F = -\frac{\hbar a}{32\pi} \cdot \frac{\omega_p^2\omega_{p0}^2}{AB_0(A+B_0)} \cdot \frac{1}{d^4} \tag{34}$$

(where $\hbar$ is Planck's constant). We note that in such classical calculations the van der Waals-London force arises from the delocalized energy of the static electromagnetic field acting upon the cross-sectional area.

Half-space and a spherical shell

We consider a spherical shell of radius a and small thickness ε, placed at the origin. According to our discussion above for the sphere, we take the displacement field as

$$\mathbf{u} = \alpha\varepsilon a(0,\, 0,\, 1)\delta(R-a) \tag{35}$$

and, within the dipole approximation, the charge density is given by $\rho = -\alpha\varepsilon a P_1(\cos\theta)\delta'(R-a)$. The potential is vanishing inside the sphere and it is given by $\Phi(\mathbf{R}) = 4\pi\alpha\varepsilon a^3 z/R^3$ outside the sphere ($R > a$); it is discontinuous at the shell, as expected. Since $z/R^3 = -\partial f/\partial z$, where $f = 1/R$ and $f(\mathbf{k}, z) = 2\pi e^{-k|z|}/k$, we get easily the field produced by the shell in the region of the half-space (the external field for the half-space), $E_{01} = -8\pi^2 i\alpha\varepsilon a^3 k e^{-kz}$, so that equation (17) becomes

$$(\omega^2 - \omega_c^2 - \frac{1}{2}\omega_p^2)A_1 = 2\pi i\omega_{p0}^2\alpha\varepsilon a^3 k \ . \tag{36}$$

We assume that the shell has not an (internal) polarization (this may be a rather unrealistic condition, which can be removed by specific calculations for a shell of finite thickness). Equation (1) becomes

$$(\omega^2 - \omega_{c0}^2)\alpha\varepsilon a\delta(R-a) = -\frac{q}{m}E_{0z} \ , \tag{37}$$

where E_{0z} is the field created by the half-space in the region of the shell. Its Fourier transform, according to equations (16), is $E_{0z}(\mathbf{k},\, z) = 2\pi i A_1 e^{k(z-2d)} \simeq 2\pi i A_1 e^{-2kd}$ ($z \ll d$). The Fourier transform of the function $\delta(R-a)$ is $\simeq 2\pi a$, and we get, from equation (37),

$$(\omega^2 - \omega_{c0}^2)\alpha\varepsilon a^2 = -\frac{i}{4\pi}\omega_p^2 A_1 e^{-2kd} \ . \tag{38}$$

By equations (36) and (38) we get the dispersion equation

$$(\omega^2 - \omega_c^2 - \frac{1}{2}\omega_p^2)(\omega^2 - \omega_{c0}^2) = \frac{1}{2}\omega_p^2\omega_{p0}^2 a k e^{-2kd} \tag{39}$$

(which, noteworthy, does not depend on the thickness ε). Up to a numerical factor, this equation is identical with equation (30) for a half-space and a sphere. However, there appear here some particularities. For $\omega_{c0} \neq 0$ the force is given by

$$F = -\frac{3\hbar a}{64\pi}\cdot\frac{\omega_p^2\omega_{p0}^2}{\omega_{c0}A(A+\omega_{c0})}\cdot\frac{1}{d^4} \ , \tag{40}$$

while for $\omega_{c0} = 0$ ("conducting" shell), the force is

$$F = \frac{3\hbar a}{64\pi}\cdot\frac{\omega_p^2\omega_{p0}^2}{A^3}\cdot\frac{1}{d^4} \ ; \tag{41}$$

it is a repulsive force. This is due to the absence of the (internal) polarization of the shell.

Half-space and a point-like body

The displacement field of a point-like particle placed at the origin can be written as

$$\mathbf{u} = a^3\mathbf{u}_0\delta(\mathbf{R}) \ , \tag{42}$$

where $\mathbf{u}_0$ is a constant vector and a is a measure for the "radius" of the body. We assume that the body has a charge density $n_0 q$ (neutralized by the rigid background). This factor is left aside for the moment, but it is restored in the final formulae. The potential can be computed immediately,

$$\Phi(\mathbf{R}) = a^3 \frac{\mathbf{u}_0 \mathbf{R}}{R^3} \ , \tag{43}$$

and the field created by the body is given by

$$E_{01}(\mathbf{k}, z) = -2\pi a^3 k (u_{01} + i u_{0z}) e^{-k|z|} \tag{44}$$

(and $E_{02}(\mathbf{k}, z) = 0$). Equations (17) become

$$(\omega^2 - \omega_c^2 - \frac{1}{2}\omega_p^2) A_1 = \frac{1}{2}\omega_{p0}^2 a^3 k (u_{01} + i u_{0z}) \ . \tag{45}$$

There is no internal (polarization) field, and the equation of motion (1) can be written as

$$(\omega^2 - \omega_{c0}^2) a^3 \mathbf{u}_0 \delta(\mathbf{R}) = -\frac{q}{m}\mathbf{E}_0 \ . \tag{46}$$

for the point-like body. The Fourier transform of the function $\delta(\mathbf{R})$ is $\delta(z)$, which we approximate by $1/a$ for z "inside" the body. The Fourier transforms of the half-space field are given by equations (16), so we get

$$(\omega^2 - \omega_{c0}^2) a^2 u_{01} = \frac{1}{2}\omega_p^2 A_1 e^{-2kd} \ , \ (\omega^2 - \omega_{c0}^2) a^2 u_{0z} = -\frac{i}{2}\omega_p^2 A_1 e^{-2kd} \ . \tag{47}$$

These equations, together with equation (45), lead to the dispersion equation

$$(\omega^2 - \omega_c^2 - \frac{1}{2}\omega_p^2)(\omega^2 - \omega_{c0}^2) = \frac{1}{2}\omega_p^2 \omega_{p0}^2 a k e^{-2kd} \ , \tag{48}$$

which is identical with the equation for the couple half-space-spherical shell. Consequently, the force between a half-space and a point-like particle is the same as the force between a half-space and a spherical shell.

Two spheres

We consider a sphere of radius a placed at the origin, with the charge density $n_0 q$, and a second sphere, of radius b, placed at $z = d$, with the charge density nq. Within the same dipole approximation (with coefficients α and, respectively, α'), we use the results described above for the sphere. For instance, for the sphere placed at the origin, equation (25) gives

$$(\omega^2 - \omega_{c0}^2 - \frac{1}{3}\omega_{p0}^2)\pi\alpha a^3 = -\frac{q}{m} E_{0z}^{(2)} \ , \tag{49}$$

where $E_{0z}^{(2)}$ is the z-component of the Fourier transform of the field produced by the second sphere at the origin. Similarly, for the second sphere we have the equation

$$(\omega^2 - \omega_c^2 - \frac{1}{3}\omega_p^2)\pi\alpha' b^3 = -\frac{q}{m} E_{0z}^{(1)} \ , \tag{50}$$

where the field $E_{0z}^{(1)}$ is given by

$$E_{0z}^{(1)} = \frac{8\pi^2}{3}\alpha a^4 k e^{-k|z|} \ , \tag{51}$$

according to equation (28), and the field produced by the second sphere (displaced by the distance d) is

$$E_{0z}^{(2)} = \frac{8\pi^2}{3}\alpha' b^4 k e^{-k|z-d|} \ . \tag{52}$$

Introducing these fields in equations (49) and (50), we get the dispersion equation

$$(\omega^2 - \omega_c^2 - \frac{1}{3}\omega_p^2)(\omega^2 - \omega_{c0}^2 - \frac{1}{3}\omega_{p0}^2) = \frac{4}{9}\omega_p^2\omega_{p0}^2 abk^2 e^{-2kd} \ . \tag{53}$$

With the notations $B^2 = \omega_c^2 + \omega_p^2/3$, $B_0^2 = \omega_{c0}^2 + \omega_{p0}^2/3$ we get the force (per unit area)

$$F = -\frac{\hbar ab}{12\pi} \cdot \frac{\omega_p^2\omega_{p0}^2}{BB_0(B+B_0)} \cdot \frac{1}{d^5} \tag{54}$$

for distinct substances ($B \neq B_0$). For identical substance, the force is given by

$$F = -\frac{\hbar ab}{24\pi} \cdot \frac{\omega_p^4}{B^3} \cdot \frac{1}{d^5} \ . \tag{55}$$

We can se that the force goes like d^{-5} (leading contribution in powers of $1/d$).

Sphere-spherical shell couple

We consider a sphere of radius a placed at the origin, with the charge density $n_0 q$, and a spherical shell of radius b and thickness ε, with the charge density nq, placed at $z = d$. The results obtained above can be transcribed immediately for this situation. The dispersion equation is

$$(\omega^2 - \omega_{c0}^2 - \frac{1}{3}\omega_{p0}^2)(\omega^2 - \omega_c^2) = \frac{4}{3}\omega_p^2\omega_{p0}^2 abk^2 e^{-2kd} \ . \tag{56}$$

For $\omega_c \neq 0$ the force is given by

$$F = -\frac{\hbar ab}{4\pi} \cdot \frac{\omega_p^2\omega_{p0}^2}{\omega_c B_0(B_0+\omega_c)} \cdot \frac{1}{d^5} \ , \tag{57}$$

with B_0 defined above. For a "conducting" shell ($\omega_c = 0$) the force is repulsive,

$$F = \frac{\hbar ab}{4\pi} \cdot \frac{\omega_p^2\omega_{p0}^2}{B_0^3} \cdot \frac{1}{d^5} \ . \tag{58}$$

Two spherical shells

Similarly, the results described above can be used for two spherical shells. With the same notations and conventions as above, the dispersion equation for two spherical shells is given by

$$(\omega^2 - \omega_{c0}^2)(\omega^2 - \omega_c^2) = \omega_p^2\omega_{p0}^2 abk^2 e^{-2kd} \ . \tag{59}$$

For ω_c, $\omega_{c0} \neq 0$ we get the force

$$F = -\frac{3\hbar ab}{16\pi} \cdot \frac{\omega_p^2\omega_{p0}^2}{\omega_c\omega_{c0}(\omega_c+\omega_{c0})} \cdot \frac{1}{d^5} \ . \tag{60}$$

If one shell is "conducting" (say $\omega_{c0} = 0$), the force is repulsive, given by

$$F = \frac{3\hbar ab}{16\pi} \cdot \frac{\omega_p^2\omega_{p0}^2}{\omega_c^3} \cdot \frac{1}{d^5} \ . \tag{61}$$

An interesting situation occurs if both shells are "conducting" ($\omega_c = \omega_{c0} = 0$). In this case the solution of the dispersion equation is $\omega = (ab)^{1/4}\sqrt{\omega_p\omega_{p0}}k^{1/2}e^{-kd/2}$, and the (repulsive) force is given by

$$F = \frac{15\hbar(ab)^{1/4}\sqrt{\omega\omega_{p0}}}{4\sqrt{2\pi}} \cdot \frac{1}{d^{7/2}} \ . \tag{62}$$

Combinations with a point-like body. For the couple sphere-point-like body the dispersion equation is given by

$$(\omega^2 - \omega_{c0}^2 - \frac{1}{3}\omega_{p0}^2)(\omega^2 - \omega_c^2) = \frac{2}{3}\omega_p^2\omega_{p0}^2 abk^2e^{-2kd} \ ; \tag{63}$$

it is the same as equation (56), except for a factor 2/3 instead of 4/3 in the *rhs* of this equation. The force can be deduced easily from the results for the couple sphere-spherical shell. It goes like $1/d^5$, and it is repulsive for a "conducting" point-like body. Similarly, the couple spherical shell-point-like body is governed by the dispersion equation corresponding to a pair of spherical shells.

Finally, we consider two point-like bodies, one, with "radius" a and charge density n_0q, placed at the origin, another, with "radius" b and charge density nq, placed at $z = d$. We use displacements as those given by equation (42), with two distinct constant vectors $\mathbf{u}_0$ and $\mathbf{v}_0$. The calculations are similar with those described above for one point-like body placed at the origin. The difference comes from the second body, displaced by the distance d. The field generated by this body is similar with the field given by equation (44), where $e^{-k|z|}$ is replaced by $e^{-k|z-d|}$. The dispersion equation is the same as equation (59) for two spherical shells.

Concluding remarks

We have presented in this paper a series of computations regarding the classical interaction of the electromagnetic field with macroscopic, polarizable bodies, in the non-retarded regime (quasi-static field), and derived the classical counterpart of the van der Waals-London forces. Specifically, the calculations were carried out for any couple of the following bodies: one half-space (a semi-infinite solid with a plane surface) and spheres, spherical shells and point-like bodies. The polarization is represented by a displacement field subjected to the classical (Newton) equation of motion. The coupled equations of motion for two interacting bodies are solved for the eigenfrequencies, and the energy is estimated as the correction, due to the interaction, to the zero-point energy of the field (vacuum fluctuations). The force is derived from the variation of this energy with the separation distance d. We have limited ourselves to the non-retarded regime, corresponding to the van der Waals-Lndon forces (small separation distances), a situation which is of interest experimentally. The dipole approximation has been used for the spherical bodies (which implies small dimensions of the bodies in comparison with the separation distance).

The van der Waals-London force for a half-space and any other body considered here goes like $1/d^4$. The force acting between two spheres goes like $1/d^5$. These forces are attractive. The spherical shells and the point-like bodies behave very much alike the spheres, except for the absence of an (internal) polarization. This, rather unrealistic, feature may give rise to some particularities, especially where such bodies are "conducting". For instance, a "conducting" shell, or a "conducting" point-like body coupled to any other body gives rise to a repulsive force. If both shells, or point-like bodies are "conducting", the force is repulsive and goes like $1/d^{7/2}$. A great variety of interacting, polarizable, compact bodies can be investigated by the method presented here (for instance, polarizable spherical shells of finite thickness), with a large variety of results regarding the classical counterpart of the van der Waals-London force acting between macroscopic bodies. The retarded regime, for the classical counterpart of the Casmir force, can be treated similarly. The corresponding results will be presented in a forthcoming publication.

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References

[1] H. B. G. Casimir and D. Polder, "The influence of retardation on the London-van der Waals forces," Phys. Rev. **73** 360-372 (1948).

[2] H. Casimir, "On the atttraction between two perfectly conducting plates," Proc. Kon. Ned. Ak. Wet. **51** 793 (1948).

[3] R. Eisenschitz and F. London, "Uber das Verhaltnis der van der Waalsschen Krafte zu den homoopolaren Bindungskraften", Z. Physik **60** 491-527 (1930).

[4] F. London, "Zur Theorie und Systematik der Molekularkrafte", Z. Physik **63** 245-279 (1930).

[5] F. London, "The general theory of molecular forces", Trans. Faraday Soc. **33** 8-26 (1937).

[6] E. Lifshitz, ZhETF **29** 94-105 (1956) (Sov. Phys. JETP **2** 73-83 (1956)).

[7] I. E. Dzyaloshinskii, E. M. Lifshitz and L. P. Pitaevskii, "The general theory of van der Waals forces," Adv. Phys. **10** 165-209 (1961).

[8] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 5 (*Statistical Physics*), part. 2, Butterworth-Heinemann, Oxford (2003).

[9] J. Schwinger, "Casimir effect in source theory", Lett. Math. Phys. **1** 43-47 (1975).

[10] J. Schwinger, L. L. deRaad, Jr., and K. A. Milton, "Casimir effect in dielectrics", Ann. Phys. **115** 1-23 (1978).

[11] N. G. Van Kampen, B. R. A. Nijboer and K. Schram, "On the macroscopic theory of van der Waals forces," Phys. Lett. **A26** 307-308 (1968).

[12] T. H. Boyer, "Recalculations of long-range van der Waals potentials", Phys. Rev. **180** 19-24 (1969).

[13] E. Gerlach, "Equivalence of van der Waals forces between solids and the surface-plasmon interaction," Phys. Rev. **B4** 393-396 (1971).

[14] K. Schram, "On the macroscopic theory of retarded van der Waals forces," Phys. Lett. **A43** 282-284 (1973).

[15] J. Heinrichs, "Theory of van der Waals interaction between metal surfaces," Phys. Rev. **B11** 3625-3636 (1975).

[16] P. W. Milloni, *The Quantum Vacuum* (Academic Press, San Diego, 1994).

[17] V. M. Mostepanenko and N. N. Trunov, *The Casimir Effect and its Applications* (Clarendon, Oxford, 1997).

[18] S. K. Lamoreaux, "Demonstration of the Casimir force in the 0.6 to $6\mu m$ range," Phys. Rev. Lett. **78** 5-8 (1997).

[19] L. H. Ford, "Casimir sphere between a dielectric sphere and a wall: a model for amplification of vacuum fluctuations, Phys. Rev. **A58** 4279-4286 (1998).

[20] U. Mohideen and A. Roy, "Precision measurement of the Casimir force from 0.1 to $0.9\mu m$", Phys. Rev. Lett. **81** 4549-4552 (1998).

[21] S. K. Lamoreaux, "Calculation of the Casimir force between imperfectly conductiong plates," Phys. Rev. **A59** R3149-R3153 (1999).

[22] A. Lambrecht and S. Reynard, "Comment on "Demonstration of the Casimir force in the 0.6 to $6\mu m$ range"," Phys. Rev. Lett. **84** 5672-5672 (2000).

[23] A. Lambrecht and S. Reynaud, "Casimir force between metallic mirrors", Eur. Phys. J. **D8** 309-318 (2000).

[24] M. Bordag, U. Mohideen and V. M. Mostepanenko, "New developments in the Casimir effect," Phys. Reps. **353** 1-205 (2001).

[25] K. A. Milton, *The Casimir Effect* (World Scientific, Singapore, 2001).

[26] G. Bressi, G. Carugno, R. Onofrio and G. Ruoso, "Measurement of the Casimir force between parallel metallic surfaces", Phys. Rev. Lett. **88** 041804 (2002) (1-4).

[27] C. Genet, A. Lambrecht and S. Reynaud, "Casimir force and the quantum theory of lossy optical cavities," Phys. Rev. **A67** 043811 (1-18) (2003).

[28] K. A. Milton, "The Casimir effect: recent controversies and progress", J. Phys. A: Math. Gen. **37** R209-R277 (2004).

[29] F. Chen, U. Mohideen, G. L. Klimchitskaya and V. M. Mostepanenko, "Investigation of the Casimir force between metal and semiconductor test bodies," Phys. Rev. **A72** 020101(R1-4) (2005).

[30] S. K. Lamoreaux, "The Casimir force: background, experiments and applications," Reps. Progr. Phys. **65** 201-236 (2005).

[31] F. Intravaia, "Effet Casimir et interaction entre plasmons de surface," These de Doctorat de l'Universite Paris VI, 1-177, June 2005.

[32] J. M. Obrecht, R. J. Wild, M. Antezza, L. P. Pitaevskii, S. Stringari and E. A. Cornell, "Measurement of the temperature dependence of the Casimir-Polder force," Phys. Rev. Lett. **98** 063201 (1-4) (2007).

[33] F. Intravaia, C. Henkel and A. Lambrecht, "Role of surface plasmons in the Casimir effect," Phys. Rev. **A76** 033820 (1-11) (2007).

[34] T. Emig, N. Graham, R. L. Jaffe and M. Kardar, "Casimir forces between arbitrary compact objects", Phys. Rev. Lett. **99** 170403 (2007) (1-4).

[35] M. Kruger, T. Emig, G. Bimonte and M. Kardar, "Non-equilibrium Casimir forces:: spheres and sphere-plate", Eur. Phys. Lett. **95** 21002 (2011) (1-6).

[36] M. Apostol and G. Vaman, "Electromagnetic eigenmodes in matter. van der Waals-London and Casimir forces", Pogr. Electrom. Res. PIER **B19** 115-131 (2010).

[37] M. Apostol and G. Vaman, "Attraction force between a polarizable point-like particle and a semi-infinite solid", Roum. J. Phys. **55** 764-771 (2010).

[38] P. Drude, "Zur Elektronentheorie der Metalle", Ann. Phys. **306** 566-613 (1900).

[39] P. Drude, "Zur Elektronentheorie der Metalle, 2. Teile. Galvanomagnetische und thermomagnetische Effecte, **308** 369-402 (1900).

[40] H. A. Lorentz, *The Theory of Electrons*, Teubner, Leipzig (1916).

[41] N. W. Ashcroft and N. D. Mermin, *Solid State Physics*, Saunders College,Philadelphia (1976).

[42] G. D. Mahan, *Many-Particle Physics*, Plenum, NY (1980).

[43] H. Schubert and B. Wilhelmi, *Nonlinear Optics and Quantum Electronics*, Wiley, NY (1986).

[44] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, eds. A. Jeffrey and D. Zwillinger, 6th edition, Academic Press, San Diego (2000), pp. 714-715, 6.677.1,2.

[45] M. Apostol and G. Vaman, "Electromagnetic field interacting with a semi-infinite plasma", J. Opt. Soc. Am. **A26** 1747-1753 (2009).

[46] M. Apostol and G. Vaman, "Plasmons and difraction of an electromagnetic plane wave by a metallic sphere", Progr. Electrom. Res. PIER **98** 97-118 (2009).

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