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A new derivation of the WKB approximation

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Abstract

A new derivation of the WKB approximation is provided, based on the wave propagation in a slightly inhomogeneous medium.

The WKB approximation (or WKBJ approximation, after the names of Wentzel, Kramers, Brillouin and Jeffreys, see for example Ref.1) provides the solution of the second-order differential equation

$$\frac{d^2\psi}{dx^2} = -k^2(x)\psi \tag{1}$$

with $k(x)$ (in general complex) a slowly varying function of x. With the substitution

$$\psi(x) = e^{\Phi(x)} \tag{2}$$

the function $\Phi(x)$ satisfies the equation

$$\Phi'^2(x) + \Phi''(x) + k^2(x) = 0 \quad , \tag{3}$$

whence

$$\Phi'^2(x) = -k^2(x) \pm ik'(x) + ... \quad . \tag{4}$$

Introducing (4) into (2) we obtain the WKB solution

$$\psi(x) = \frac{C}{\sqrt{k(x)}} e^{\pm i\int_{x_0}^{x} k\cdot dx} \quad , \tag{5}$$

where C and x_0 are constants of integration.The function $k(x)$ can be viewed as a wavevector and $\lambda(x) = 1/k(x)$ can be taken as a wavelength; the solution given by (5) is then valid for

$$\left|\frac{d\lambda}{dx}\right| \ll 1 \quad . \tag{6}$$

The WKB approximation, also known as the semi-classical approximation in quantum mechanics, and the geometrical optics approximation in wave theory,[2] has been studied extensively, especially by Langer.[3] We present in this note a new derivation of it, suuggested by the wave propagation in a slightly inhomogeneous medium.

Let us assume that a wave of unit amplitude and of wavevector k_0 propagates through a slightly inhomogeneous medium from $x_0 = 0$ to $x_N = L$. The wave is reflected at $x_0 = 0$ with the amplitude R, and is transmitted at the other end $x_N = L$ with the amplitude T. The propagation is assumed to proceed along a straight line and with a wavevector $k(x)$ which depends slightly on position, *i.e.* the variation of the corresponding wavelength over the scale length of the medium inhomogeneities is much smaller than unity. This is precisely the condition of geometrical optics given by (6). Accordingly, we divide the medium sample into slices of coordinates x_n, $n = 0, 1, ...N$, such that the wavevector $k(x) = k_n$ is constant for $x_{n-1} < x < x_n$; within each of these intervals we may write the wave as

$$\psi_n(x) = A_n e^{ik_n x} + B_n e^{-ik_n x} \ , \ x_{n-1} < x < x_n, \ n = 0, 1, ...N + 1 \quad , \tag{7}$$

where $A_0 = 1, \ B_0 = R, \ A_{N+1} = T, \ B_{N+1} = 0$, x_{-1} and x_{N+1} being arbitrary. A schematic picture of this setup is shown in Fig.1.

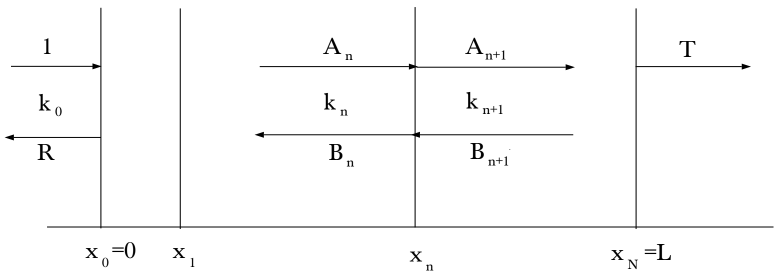

Figure 1: A schematic picture of the setup used in text for wave propagation in an inhomogeneous medium

The continuity conditions

$$\psi_n(x_n) = \psi_{n+1}(x_n) \quad , \quad \psi_n^{'}(x_n) = \psi_{n+1}^{'}(x_n) \tag{8}$$

allow one to express the amplitudes $A_{n+1}, \ B_{n+1}$ in terms of $A_n, \ B_n$. In matricial notation we have

$$\left(\begin{array}{c} A_{n+1} \\ B_{n+1} \end{array} \right) = M_n \left(\begin{array}{c} A_n \\ B_n \end{array} \right) \ , \ n = 0, 1, ...N \quad , \tag{9}$$

where

$$M_n = \left(\begin{array}{cc} \frac{1}{2}\left(1 + \frac{k_n}{k_{n+1}}\right) e^{-i(k_{n+1}-k_n)x_n} & \frac{1}{2}\left(1 - \frac{k_n}{k_{n+1}}\right) e^{-i(k_{n+1}+k_n)x_n} \\ \frac{1}{2}\left(1 - \frac{k_n}{k_{n+1}}\right) e^{i(k_{n+1}+k_n)x_n} & \frac{1}{2}\left(1 + \frac{k_n}{k_{n+1}}\right) e^{i(k_{n+1}-k_n)x_n} \end{array} \right) \quad . \tag{10}$$

By reiterating the relationship (9) we obtain

$$\left(\begin{array}{c} T \\ 0 \end{array} \right) = M \left(\begin{array}{c} 1 \\ R \end{array} \right) \quad , \tag{11}$$

where

$$M = M_N M_{N-1} ... M_1 M_0 \ . \tag{12}$$

Since k_n varies slowly with n we may limit ourselves to the first two terms in the series expansion

$$k_{n+1} = k_n + k_n^{'} + ... = k_n(1 + u_n) \quad , \tag{13}$$

where

$$u_n = \frac{d}{dn}\left(\ln k_n\right) \ll 1 \ . \tag{14}$$

Using this linear approximation in (10) we can express the matrix M_n as

$$M_n = M_n^0 \left(1 + M_n^1\right) \quad , \tag{15}$$

where

$$M_n^0 = \begin{pmatrix} e^{-u_n k_n x_n} & \\ & e^{iu_n k_n x_n} \end{pmatrix} \tag{16}$$

is a diagonal matrix and

$$M_n^1 = \frac{1}{2} u_n \begin{pmatrix} -1 & e^{-2ik_n x_n} \\ e^{2ik_n x_n} & -1 \end{pmatrix} \quad . \tag{17}$$

Within the same approximation we can compute now the product of matrices in (12) as

$$M = M_N^0 M_{N-1}^0 ... M_1^0 M_0^0 \left(1 + M_N^1 + M_{N-1}^1 + ... + M_1^1 + M_0^1\right) \quad , \tag{18}$$

and obtain

$$M = \begin{pmatrix} (1-u/2)\, e^{-i\varphi} & \varepsilon e^{-i\varphi} \\ \varepsilon^* e^{i\varphi} & (1-u/2)\, e^{i\varphi} \end{pmatrix} \quad , \tag{19}$$

with the following notations:

$$u = \sum_{n=0}^{N} u_n = \int_0^N dn \cdot \frac{d}{dn} (\ln k_n) = \ln\left[k(L)/k_0\right] \quad , \tag{20}$$

$$\varphi = \sum_{n=0}^{N} u_n k_n x_n \int_0^N dn \cdot k_n' x_n = \int_0^L dk \cdot x = k(L)L - \int_0^L k \cdot dx \quad , \tag{21}$$

$$\varepsilon = \sum_{n=0}^{N} u_n e^{-2ik_n x_n} \quad . \tag{22}$$

In the above equations we have replaced the summations by integrations since k_n is a slowly varying function of n. It is easy to see now that ε given by (22) can be neglected within the present approximation. Indeed, we have the following estimation for ε:

$$\varepsilon = \sum_{n=0}^{N} u_n e^{-2ik_n x_n} = \int_0^L dx \cdot (\ln k)' \, e^{-2ikx} \simeq u \cdot \frac{e^{-2ik(L)L} - 1}{-2ik(L)L} \quad , \tag{23}$$

and we can see that ε is smaller than u by a factor of order $1/k(L)L \ll 1$. In other words, the interference effects in this short-wavelength approximation lead to a vanishing reflection coefficient R, because, making use of (11) and (19), we obtain

$$R = -\frac{\varepsilon^*}{1 - u/2} \simeq 0 \tag{24}$$

and

$$T \cong (1-u/2)\, e^{-i\varphi} = \left[1 - \frac{1}{2}\ln\left[k(L)/k_0\right]\right] \cdot e^{-ik(L)L + i\int_0^L k \cdot dx} \quad . \tag{25}$$

The outgoing wave at $x_N = L$ is therefore

$$\psi(L) = \left[1 - \frac{1}{2}\ln\left[k(L)/k_0\right]\right] \cdot e^{i\int_0^L k \cdot dx} \quad , \tag{26}$$

and we can see that its amplitude is the sum of the first two terms in the expansion of $[k_0/k(L)]^{1/2}$; according to (5) this is precisely the WKB solution. It is easy to check that the "current" conservation $\psi^* (\partial\psi/\partial x) - (\partial\psi^*/\partial x)\,\psi = const$ amounts to $k_0 \left(1 - |R|^2\right) = k(L)\,|T|^2$, which is verified by (24) and (25).

The above calculations are presented in terms of purely propagating waves, *i.e.* without absorption, as if the wavevector $k(x)$ were purely real; actually, the calculations remain valid for a complex $k(x)$. Indeed, assuming that $k(x)$ acquires an imaginary part $\kappa(x)$ the wave $\psi(x)$ in (7) must be replaced by

$$\psi_n(x) = A_n e^{ik_n x} e^{-\kappa_n(x-x_{n-1})} + B_n e^{-ik_n x} e^{-\kappa_n(x_n - x)}\ ,\ x_{n-1} < x < x_n \quad , \tag{27}$$

such as to account for absorption. Following the same reasoning as above we arrive again at equation (26) with a complex $k(x)$.

References

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