Journal of Theoretical Physics

Founded and Edited by M. Apostol **200** (2011)

ISSN 1453-4428

The electromagnetic force (van der Waals-London-Casimir force) for spheres, point-like bodies and a half-space

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Abstract

We investigate here the classical electromagnetic coupling between a sphere and a half-space (a semi-infinite solid with a plane surface), both in the retarded (radiation) and non-retarded regime. It is found that the Casimir force (retarded regime) goes like $1/d^2$ (at long distance), where d is the separation distance between the sphere and the half-space, while in the non-retarded regime the van der Waals-London force goes like $1/d^4$. Similarly, we compute also the forces betwen two spheres, two point-like bodies and a sphere and a point-like body. For identical substance, the Casimir force between two spheres is the same as for the couple sphere-half-space, while for distinct substances the Casmir force goes like $1/d^4$. In the non-retarded regime, the force between two spheres goes like $1/d^5$. The point-like bodies behaves very much alike the spheres. All the forces are attractive, except for some particular situations (related to the absence of an (internal) polarization). Spherical shells are also discussed. The calculations are performed within the well-known Lorentz-Drude (plasma) model of polarizable matter, by using the dipole approximation for the sphere. The coupled equations of motion of the polarization are obtained and solved for the electromagnetic eigenfrequencies. The force is estimated from the correction brought by interaction to the zero-point energy of the polarization field (vacuum fluctuations). The connection between the present classical theory for macroscopic bodies and the quantum-mechanical treatment is discussed.

PACS: 03.50.De; 41.20.-q; 31.30.J; 34.35.+a; 42.50.Wk

Keywords: Electromagnetic forces; Casimir force; van der Waals-London force; vacuum fluctuations

Introduction

The original derivation of the Casimir force[1, 2] has been done for the quantum interaction between a neutral atom and a half-space (a semi-infinite solid with a plane surface), two neutral atoms, as well as for two half-spaces. For small distances of separation d, the classical, non-retarded van der Waals-London results[3]-[5] have been re-obtained: the force goes like $1/d^4$ between an atom and a half-space, and like $1/d^7$ between two atoms. As it is well-known, a d^{-n}- force between two particles results in a d^{-n+3}-between a particle and a half-space (and a d^{-n+4}-force between two half-spaces). In the retarded regime (for longer distances), the (Casimir) force goes like $1/d^5$ for the couple atom-half-space and like $1/d^8$ for two atoms. All these forces are attractive. These results have been re-derived within the framework of the general quantum-statistical theory of the electromagnetic fluctuations (particularly for two interacting half-spaces),[6]-[8] and from source theory.[9, 10] The necessity for a classical treatment of the van der Waals-London-Casimir forces acting between macroscopic bodies have been emphasized in a series of subsequent papers.[11]-[16] The macroscopic bodies brought their own particularities with respect to the electrical polarization, like specific electromagnetic modes (plasmons, polaritons, surface effects, etc), as compared with quantum-mechanical behaviour of individual particles. On the other side, the actual experimental interest lies in the electromagnetic forces acting betwen macroscopic bodies, especially of finite size, like the couples sphere-half-space, two spheres, etc.[17]-[36] Therefore, a classical counterpart of the (quantum) van der Waals-London and Casimir forces is expected.

Recently, a method suitable for macroscopic bodies has been put forward,[37] based on the classical interaction between the electromagnetic field and matter. It consists mainly in representing the matter polarization by a displacement field $\mathbf{u}(\mathbf{R},t)$ of the mobile charges, function of position $\mathbf{R}$ and time t, subjected to the classical (Newton) equation of motion. The well-known Lorentz-Drude (plasma) model is employed for the polarizable (non-magnetic) matter. The electromagnetic coupling between two bodies amounts to solving the coupled equations of motion of the polarization for the eigenfrequencies. The energy is then estimated as the correction brought by interaction to the zero-point energy of the vacuum (the temperature effects, usually small, may be included) and the force is thereby derived from the variation of the energy with the separation distance bewteen the bodies. The Casimir force ($\sim 1/d^4$) acting between two half-spaces has been derived by this method, as well as the van der Waals-London force ($\sim 1/d^3$). An attempt has also been made[38] for the non-retarded van der Waals-London coupling between a point-like body and a half-space (the force going like $1/d^4$). It was shown in Ref. [37] that the Casimir force implies electromagnetic modes propagating between the two half-spaces, coupled to surface plasmon-polariton modes inside the half-spaces, the latter being propagating modes along the in-plane directions (parallel to the surface of the bodies) and damped along the direction perpendicular to the surface. We give here the results for the Casimir force ($\sim 1/d^2$) acting at large distance between a (macroscopic) sphere and a half-space, as well as the corresponding van der Waals-London force ($\sim 1/d^4$). The result $1/d^2$ can be viewed as being the Coulomb force arising between the sphere and its image in the half-space. It is worth noting that for the non-retarded regime the force is the same as for a quantum particle, while for the retarded regime, the macroscopic sphere looks like an infinite medium of quatum-mechanical particles and the relationship $d^{-n+3} - d^{-n}$ holds. Similarly, we give here the results for two interacting spheres, which exhibit particularities of macroscopic bodies (for instance, the force is different for identical or distinct substances). For identical substance, the Casimir force is the same as for the couple sphere-half-space, which, with the image-force interpretation, is not an unexpected result. For distinct substances, the Casimir force between two spheres goes like $1/d^4$, which implies, with respect to the quantum-mechanical result ($\sim 1/d^8$), that the two macroscopic spheres act like infinite media (the relationship $d^{-n+4} - d^{-n}$). In the non-retarded limit the van der Waals-London force acting between two spheres goes like $1/d^5$, and it has no quantum-mechanical analog. Coupling involving point-like bodies or spherical shells is also discussed. Such bodies behave mainly as spheres. In some special cases, repulsive forces are obtained.

The electric polarization of the material bodies can be represented as slight oscillatory movements of mobile charges with respect to a neutralizing (quasi-) rigid background. Such movements can be described by a displacement field $\mathbf{u}(\mathbf{R},t)$, as discussed above. The velocity $\mathbf{v} = \dot{\mathbf{u}}$ of the mobile charges in matter is much smaller than the light velocity c, $v/c \ll 1$, so that $\omega u/c \ll 1$, where ω is the frequency of both the oscillatory motion of the displacement $\mathbf{u}$ and the electromagnetic field (the polarization field) produced by the motion of the charges. This inequality means that matter polarization proceeds mainly by rather limited displacements u, depending on frequencies. For finite-size bodies, there is a natural limitation for such displacements, the (linear) size a

of the body. By analogy with the "dipole radiation", the condition $\omega a/c \ll 1$ can be called "dipole approximation" (corresponding to long wavelengths). We adopt this approximation here for the sphere (which implies certain limitations on the frequencies). We shall see that such an approximation amounts to estimating the leading contributions to the (retarded) forces, higher-order contributions (which would relieve the limitations imposed upon the frequencies) resulting in higher-order corrections to the force.

Lorentz-Drude model

The well-known Lorentz-Drude model of (homogeneous) polarizable matter[39]-[43] consists of identical charges q, with mass m and density n, moving in a rigid neutralizing background. A slight displacement field $\mathbf{u}(\mathbf{R}, t)$ of the mobile charges is subjected to the equation of motion

$$m\ddot{\mathbf{u}} = q(\mathbf{E} + \mathbf{E}_0) - m\omega_c^2\mathbf{u} - m\gamma\dot{\mathbf{u}} \ , \tag{1}$$

where $\mathbf{E}$ is the electric field produced by the polarization charges and currents, $\mathbf{E}_0$ is an external electric field, ω_c is a characteristic frequency of the body and γ is a damping factor. For dielectrics $\omega_c \neq 0$, for conductors $\omega_c = 0$. Since the polarization is given by $\mathbf{P} = nq\mathbf{u}$, it is easy to see that equation (1) leads to the well-known electric susceptibility

$$\chi = -\frac{\omega_p^2}{4\pi} \cdot \frac{1}{\omega^2 - \omega_c^2 + i\omega\gamma} \tag{2}$$

and dielectric function $\varepsilon = 1 + 4\pi\chi$, where $\omega_p = \sqrt{4\pi nq^2/m}$ is the plasma frequency. We note the absence of the Lorentz force in equation (1), whose contribution is quadratic in the displacement field $\mathbf{u}$ and, consequently, it may be neglected. However, we can include an external magnetic field in the equation of motion (1), if necessary.

The displacement field $\mathbf{u}$ produces polarization charge and current densities given by

$$\rho = -div\mathbf{P} = -nq div\mathbf{u} \ , \ \mathbf{j} = \frac{\partial \mathbf{P}}{\partial t} = nq\dot{\mathbf{u}} \ , \tag{3}$$

which can be used to compute the electromagnetic potentials

$$\begin{gathered} \Phi(\mathbf{R}, t) = \int d\mathbf{R}' \frac{\rho(\mathbf{R}', t - |\mathbf{R} - \mathbf{R}'|/c)}{|\mathbf{R} - \mathbf{R}'|} \ , \\ \mathbf{A}(\mathbf{R}, t) = \frac{1}{c} \int d\mathbf{R}' \frac{\mathbf{j}(\mathbf{R}', t - |\mathbf{R} - \mathbf{R}'|/c)}{|\mathbf{R} - \mathbf{R}'|} \end{gathered} \tag{4}$$

(subjected to the Lorenz gauge $div\mathbf{A} + (1/c)\partial\Phi/\partial t = 0$). These potentials give rise to the electric field $\mathbf{E}$ in equation (1), whence we can get the displacement $\mathbf{u}$. This way, we can compute the electromagnetic fields of a polarizable body, subjected to the action of an external electromagnetic field. The external fields in equation (1) are the mutual fields by which the bodies act one upon another.

Half-space

For a half-space extending over the region $z > d$ we take the polarization as

$$\mathbf{P} = nq(\mathbf{u}, \, u_z)\theta(z - d) \ , \tag{5}$$

where $\theta(z) = 0$ for $z < 0$ and $\theta(z) = 1$ for $z > 0$ is the step function, and get the polarization charge and current densities

$$\begin{gathered} \rho = -nq(div\mathbf{u} + \frac{\partial u_z}{\partial z})\theta(z - d) - nqu_z(d)\delta(z - d) \ , \\ \mathbf{j} = nq(\dot{\mathbf{u}}, \, \dot{u}_z)\theta(z - d) \ . \end{gathered} \tag{6}$$

We use the Fourier decomposition of the type

$$\mathbf{u}(\mathbf{r},z;t)=\frac{1}{(2\pi)^3}\int d\mathbf{k}d\omega\mathbf{u}(k,z;\omega)e^{-i\omega t+i\mathbf{kr}}\ \ , \tag{7}$$

and may omit ocassionally the arguments $\mathbf{k}$, ω, writing simply $\mathbf{u}(z)$, or $\mathbf{u}$. The electromagnetic potentials given by equations (4) includes the "retarded" Coulomb potential $e^{i\frac{\omega}{c}|\mathbf{R}-\mathbf{R}'|}/\left|\mathbf{R}-\mathbf{R}'\right|$, for which we use the decomposition[44]

$$\frac{e^{i\lambda|\mathbf{R}-\mathbf{R}'|}}{|\mathbf{R}-\mathbf{R}'|}=\frac{i}{2\pi}\int d\mathbf{k}\frac{1}{\kappa}e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}e^{i\kappa|z-z'|}\ \ , \tag{8}$$

where $\lambda=\omega/c$ and $\kappa=\sqrt{\lambda^2-k^2}$. The calculations are straightforward and we get the Fourier tranforms of the potentials

$$\begin{gathered}\Phi(\mathbf{k},z;\omega)==\tfrac{2\pi}{\kappa}\int_d^\infty dz'\mathbf{ku}e^{i\kappa|z-z'|}-\tfrac{2\pi i}{\kappa}\tfrac{\partial}{\partial z}\int_d^\infty dz'u_ze^{i\kappa|z-z'|}\ \ ,\\ \mathbf{A}(\mathbf{k},z;\omega)=\tfrac{2\pi\lambda}{\kappa}\int_d^\infty dz'(\mathbf{u},\,u_z)e^{i\kappa|z-z'|}\end{gathered} \tag{9}$$

(where we have left aside the factor nq; it is restored in the final formulae). In order to compute the electric field it is convenient to refer the in-plane vectors (*i.e.*, vectors parallel with the surface of the half-space) to the vectors $\mathbf{k}$ and $\mathbf{k}_\perp$, where $\mathbf{k}_\perp$ is perpendicular to $\mathbf{k}$ and of the same magnitude as $\mathbf{k}$; for instance, we write

$$\mathbf{u}=u_1\frac{\mathbf{k}}{k}+u_2\frac{\mathbf{k}_\perp}{k} \tag{10}$$

and a similar representation for the electric field parallel with the surface of the half-space. In performing the calculations, it is worth paying attention to the correct derivative of the modulus function, according to the equation

$$\frac{\partial^2}{\partial z^2}e^{i\kappa|z-z'|}=-\kappa^2e^{i\kappa|z-z'|}+2i\kappa\delta(z-z')\ . \tag{11}$$

We get the electric field

$$\begin{gathered}E_1=2\pi i\kappa\int_d^\infty dz'u_1e^{i\kappa|z-z'|}-\tfrac{2\pi k}{\kappa}\tfrac{\partial}{\partial z}\int_d^\infty dz'u_ze^{i\kappa|z-z'|}\ \ ,\\ E_2=\tfrac{2\pi i\lambda^2}{\kappa}\int_d^\infty dz'u_2e^{i\kappa|z-z'|}\ \ ,\\ E_z=-\tfrac{2\pi k}{\kappa}\tfrac{\partial}{\partial z}\int_d^\infty dz'u_1e^{i\kappa|z-z'|}+\tfrac{2\pi ik^2}{\kappa}\int_d^\infty dz'u_ze^{i\kappa|z-z'|}-4\pi u_z\theta(z-d)\ .\end{gathered} \tag{12}$$

We can check easily the equalities

$$ikE_1+\frac{\partial E_z}{\partial z}=-4\pi\left(iku_1+\frac{\partial u_z}{\partial z}\right)\theta(z-d)-4\pi u_z(d)\delta(z-d)\ \ , \tag{13}$$

which is Gauss's law, and

$$k\frac{\partial E_1}{\partial z}+i\kappa^2E_z=-4\pi i\lambda^2u_z\theta(z-d)\ \ , \tag{14}$$

which reflects the Faraday's and Maxwell-Ampere's equations. From equation (13), we can check the transversality condition $div\mathbf{E}=0$ for the electric field outside the half-space ($z<d$).

We use now the equations of motion (1) (with $\gamma=0$) for the combinations $iku_1+\partial u_z/\partial z$ and $k\partial u_1/\partial z+i\kappa^2 u_z$ in the region $z>d$. Taking into account that $div\mathbf{E}_0=0$ and $k\partial E_{01}/\partial z+i\kappa^2 E_{0z}=0$ (for a plane wave) these equations lead to

$$\frac{\partial^2 u_{1,2}}{\partial z^2}+\kappa'^2 u_{1,2}=0 \ , \tag{15}$$

where

$$\kappa'^2=\kappa^2-\frac{\lambda^2\omega_p^2}{\omega^2-\omega_c^2} \ . \tag{16}$$

We can see that $u_{1,2}=A_{1,2}e^{i\kappa' z}$, where $A_{1,2}$ are constants, *i.e.* the field propagates in the half-space with a modified wavevector κ', according to the Ewald-Oseen extinction theorem.[45] Similarly, the equations of motion (1) lead to $u_z=-(k/\kappa')A_1e^{i\kappa' z}$. The modified wavevector κ' given by equation (16) can also be written as

$$\kappa'^2=\varepsilon\frac{\omega^2}{c^2}-k^2 \ , \tag{17}$$

where $\varepsilon=1+4\pi\chi$ is the dielectric function (as given by equation (2)). We can check the well-known polaritonic dispersion relation $\varepsilon\omega^2=c^2K'^2$, where $\mathbf{K}'=(\mathbf{k},\kappa')$ is the wavevector.

The constants $A_{1,2}$ can be derived from the original equations (1) (for $z>d$). We get

$$\begin{aligned} &\tfrac{1}{2}A_1\omega_p^2\tfrac{\kappa\kappa'+k^2}{\kappa'(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{i\kappa z}=\tfrac{q}{m}E_{01} \ , \\ &\tfrac{1}{2}A_2\omega_p^2\tfrac{\lambda^2}{\kappa(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{i\kappa z}=\tfrac{q}{m}E_{02} \ . \end{aligned} \tag{18}$$

The external field in these equations is the field generated by the sphere (outside the sphere, in the region of the half-space). Similarly, the external field for the sphere is the field generated by the half-space in the region $z<d$. This latter field is given by equations (12):

$$\begin{aligned} E_1&=-2\pi A_1\tfrac{\kappa\kappa'-k^2}{\kappa'(\kappa+\kappa')}e^{i(\kappa+\kappa')d}e^{-i\kappa z} \ , \ z<d \ , \\ E_2&=-2\pi A_2\tfrac{\lambda^2}{\kappa(\kappa+\kappa')}e^{i(\kappa+\kappa')d}e^{-i\kappa z} \ , \ z<d \end{aligned} \tag{19}$$

and $E_z=(k/\kappa)E_1$. We can see that it is the field reflected by the half-space ($\kappa\rightarrow-\kappa$).

The sphere

We consider a sphere of radius a, with the center at the origin and with a density n_0q of mobile charges. The factor n_0q is left aside, but it will be restored in the final formuale (it gives the plasma frequency $\omega_{p0}=\sqrt{4\pi n_0q^2/m}$ of the sphere). The characteristic frequency of the sphere in equation (1) is denoted by ω_{c0}. The electromagnetic field generated by a sphere under the action of an external electromagnetic field has been derived in Ref. [46]. This is the well-known Mie's theory.[45, 47] Those results can be used here to investigate the coupling of the sphere with the half-space. However, the full, exact solution given in Refs. [46, 47] is unpracticable, on one side, and, on the other side, the coupling is governed by the long wavelength part of the full solution, corresponding to $\lambda a\ll 1$ (the dipole approximation). This approximation is also justified by the results obtained for two coupled half-spaces,[37] where the Casimir force implies surface plasmon-polariton modes, which are damped (evanescent) waves inside the two bodies. This is precisely the situation for the sphere, providing the condition $\lambda a\ll 1$ is fulfilled. Higher-order corrections to the dipole approximation can be included, resulting in corrections to the leading contributions to the force.

We derive here this relevant part of the solution by a direct approach, suggested by the half-space field which acts upon the sphere (equations (19)). First, we rewrite the field given by equations (19) as

$$E_{1,2} = -2\pi A_{1,2} f_{1,2}(k) \tag{20}$$

and $E_z = (k/\kappa)E_1$, where the functions $f_{1,2}(k)$ can easily be identified from equations (19). We need the Fourier transform of this field, written as

$$\mathbf{E}_\perp = E_1 \frac{\mathbf{k}}{k} + E_2 \frac{\mathbf{k}_\perp}{k} \tag{21}$$

for the in-plane component. We use the parametrization $\mathbf{r} = r(\sin\theta, \cos\theta)$, $\mathbf{k} = k(\cos\theta', \sin\theta')$ and $\mathbf{k}_\perp = k(-\sin\theta', \cos\theta')$ and get

$$\begin{gathered} \mathbf{E}_\perp(\mathbf{r}, z) = -iA_1 \int dk k f_1(k) J_1(kr)(\sin\theta, \cos\theta) - \\ -iA_2 \int dk k f_2(k) J_1(kr)(-\cos\theta, \sin\theta) \end{gathered} \tag{22}$$

and

$$E_z(\mathbf{r}, z) = -A_1 \int dk \frac{k^2}{\kappa} f_1(k) J_0(kr) \ , \tag{23}$$

where $J_{0,1}$ are the zeroth and, respectively, first order Bessel functions of the first kind. For $r \leq a$ (inside the sphere) and $\omega a/c \ll 1$ the wavevectors κ and κ' can be approximated by $\kappa \simeq \kappa' \simeq ik$ and the functions $f_{1,2}$ become

$$f_1(k) \simeq e^{-k(2d-z)} \ , \ f_2(k) \simeq -\frac{\lambda^2}{2k^2} e^{-k(2d-z)} \ . \tag{24}$$

Noticing that $2d - z > 0$ for $-a \leq z \leq a$ (inside the sphere), the integrals intervening in equations (22) and (23) can be found in Ref. [44], p. 686 (6.611.1) and p. 694 (6.623.3). For $d \gg a$ we get the electric field produced by the half-space inside the sphere

$$\begin{gathered} E_{0x} = -\tfrac{i}{8d^3}(A_1 x + \lambda^2 d^2 A_2 y) \ , \ E_{0y} = -\tfrac{i}{8d^3}(A_1 y - \lambda^2 d^2 A_2 x) \ , \\ E_{0z} = \tfrac{i}{4d^2} A_1 \ . \end{gathered} \tag{25}$$

The suffix 0 is attached here because this field plays the role of the external field for the sphere. Within our approximation we may leave aside the A_2-terms in equations (25).

Equations (25) suggest that the displacement field inside the sphere is of the form

$$\mathbf{u} = (\alpha x + \beta y, \, \alpha y - \beta x, \, \gamma a)\theta(a - R) \ , \tag{26}$$

where α, β and γ are constants to be determined. We compute the electromagnetic potentials Φ and $\mathbf{A}$ given by equations (4) with $\rho = -div\mathbf{u}$ and $\mathbf{j} = -i\omega\mathbf{u}$, where $\mathbf{u}$ is given by equation (26). A further simplification can be made, by noticing that the charge density can be written as

$$\rho = -2\alpha\theta(a - R) + \frac{2}{3}\alpha a\delta(a - R) + \gamma a P_1(\cos\theta)\delta(a - R) - \frac{2}{3}\alpha a P_2(\cos\theta)\delta(a - R) \ , \tag{27}$$

where $P_n(\cos\theta)$ are the Legendre polynomials and $\cos\theta = z/R$ $(= z/a)$. Within the dipole approximation, the P_2-term may be left aside. This amounts to putting $\alpha = 0$. In addition, the coefficient β can also be set equal to zero, in comparison with the z-component of the displacement (γa). This is in accordance with the observation made above regarding the absence of the A_2-terms in equations (25). Therefore, we are left with $\rho = \gamma z\delta(a-R)$ and $\mathbf{j} = -i\omega\gamma a(0,0,1)\theta(a-R)$. We use the well-known decomposition of the spherical wave (Ref. [44], p. 930, 8.533.1)

$$\frac{e^{i\lambda|\mathbf{R}-\mathbf{R}'|}}{\lambda|\mathbf{R}-\mathbf{R}'|} = i\sum\nolimits_{n=0}(2n+1)j_n(\lambda R_<)h_n(\lambda R_>)P_n(\cos\Theta) \tag{28}$$

and the addition formula (Ref. [44], p. 965, 8.814)

$$P_n(\cos\Theta) = P_n(\cos\theta)P_n(\cos\theta') + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\varphi-\varphi')\ , \tag{29}$$

where P_n^m are the associated Legendre functions, j_n and h_n are the spherical Bessel functions (of the first kind and, respectively, the Hankel functions), $R_< = \min(R,\,R')$, $R_> = \max(R,\,R')$, (θ,φ) define the direction of $\mathbf{R}$, $(\theta',\,\varphi')$ define the direction of $\mathbf{R}'$ and Θ ($\cos\Theta = \sin\theta\sin\theta'\cos(\varphi-\varphi')+\cos\theta\cos\theta'$) is the angle between $\mathbf{R}$ and $\mathbf{R}'$. The calculations are straightforward. We get the leading contributions ($\lambda a \ll 1$) for $R < a$ (inside the sphere)

$$\Phi = \frac{4\pi}{3}\gamma az\ ,\ \mathbf{A} = -2\pi i\lambda\gamma a(a^2 - \frac{1}{3}R^2)(0,\,0,\,1)\ . \tag{30}$$

We can check the Lorenz gauge $div\mathbf{A} - i\lambda\Phi = 0$. We can see also that $\mathbf{A}$ may be neglected in comparison with Φ, in the limit $\lambda a \ll 1$. In this limit, the electric field inside the sphere ($\mathbf{E} = -grad\Phi$) is given by

$$\mathbf{E} = -\frac{4\pi}{3}\gamma a(0,\,0,\,1)\ . \tag{31}$$

We introduce this field, together with the external field given by equations (19), in the equation of motion (1), which becomes

$$(\omega^2 - \omega_{c0}^2 - \frac{1}{3}\omega_{p0}^2)\gamma a\theta(a-R) = -\frac{q}{m}E_{0z}\ . \tag{32}$$

We can recognize in the *lhs* of this equation the lowest (dipole) frequency $\omega_{p0}/\sqrt{3}$ of the spherical plasmon.[46] The Fourier transform of the function $\theta(a-R)$ entering equation (32) can be computed easily. For $ak \ll 1$ it is given by πa^2, so that equation (32) becomes

$$(\omega^2 - \omega_{c0}^2 - \frac{1}{3}\omega_{p0}^2)\pi\gamma a^3 = \frac{1}{2}\omega_p^2 A_1\frac{k(\kappa\kappa' - k^2)}{\kappa\kappa'(\kappa+\kappa')}e^{i(\kappa+\kappa')d}\ . \tag{33}$$

We turn now to the field created by the sphere within the half-space. It plays the role of the external field in equations (18). For $R > a$, the leading contributions to the electromagnetic potentials in the limit $\lambda a \ll 1$ are given by

$$\Phi = \frac{4\pi i}{3}\lambda^2 a^4\gamma P_1(\cos\theta)h_1(\lambda R)\ ,\ \mathbf{A} = \frac{4\pi}{3}\lambda^2 a^4\gamma h_0(\lambda R)(0,\,0,\,1)\ . \tag{34}$$

We can check the Lorenz gauge $div\mathbf{A} - i\lambda\Phi = 0$, the wave (Helmholtz) equations $\lambda^2\Phi + \Delta\Phi = 0$, $\lambda^2\mathbf{A} + \Delta\mathbf{A} = 0$ and the transversality conditions $div\mathbf{E} = 0$ for the electric field $\mathbf{E} = i\lambda\mathbf{A} - grad\Phi$. Comparing equations (34) with equations (30), and using the asymptotic formulae $h_0(z) \simeq -i/z$, $h_1(z) \simeq -i/z^2$ for $z \ll 1$, we can check also the continuity of the potentials at $R = a$. It is easy to see that the scalar potential can also be derived from $i\lambda\Phi = \partial A_z/\partial z$, so that the Fourier transform of the in-plane field is given by

$$\mathbf{E}_\perp = -i\mathbf{k}\Phi = -\frac{1}{\lambda}\mathbf{k}\frac{\partial A_z}{\partial z}\ , \tag{35}$$

where A_z is the Fourier transform of the vector potential $(A_z(\mathbf{k}, z; \omega))$. We can see that the $\mathbf{k}_\perp$-component of the electric field is vanishing, so we have $u_2 = 0$ ($A_2 = 0$ in equations (18)). The Fourier transform of the vector potential

$$A_z = \frac{4\pi}{3}\lambda^2 a^4 \gamma \int d\mathbf{r} h_0(\lambda R) e^{-i\mathbf{kr}} \tag{36}$$

implies the Bessel function $J_0(kr)$. Making use of $h_0(z) = -ie^{iz}/z$ the integral in equation (36) acquires the form of well-known integrals given in Ref. [44], pp. 714-715, 6.677.1,2. We get

$$A_z(\mathbf{k}, z) = \frac{8\pi^2}{3}\frac{\lambda a^4}{\kappa}\gamma e^{i\kappa z} \tag{37}$$

and the electric field

$$E_1 = -\frac{8\pi^2 i}{3} a^4 \gamma k e^{i\kappa z} \ . \tag{38}$$

Making use of this (external) field, equations (18) become

$$\frac{1}{2} A_1 \omega_p^2 \frac{\kappa\kappa' + k^2}{\kappa'(\kappa' - \kappa)} e^{i(\kappa' - \kappa)d} = -\frac{2\pi i}{3}\omega_{p0}^2 a^4 \gamma k \ . \tag{39}$$

From equations (33) and (39) we get the dispersion equation

$$e^{2i\kappa d} = \frac{3i\kappa}{2ak^2\omega_{p0}^2}(\omega^2 - \omega_{c0}^2 - \frac{1}{3}\omega_{p0}^2)\frac{\kappa\kappa' + k^2}{\kappa\kappa' - k^2} \cdot \frac{\kappa' + \kappa}{\kappa' - \kappa} \ . \tag{40}$$

The solutions (ω) of this equation are the electromagnetic eigenfrequencies of the sphere coupled with the half-plane (within the dipole approximation).

Electromagnetic eigenfrequencies and the Casimir force

Equation (40) has solutions only for κ' purely imaginary, *i.e.* $\kappa' = i\alpha$, $\alpha^2 = \lambda^2\omega_p^2/(\omega^2 - \omega_c^2) - \kappa^2 > 0$. In this case, it can be written as

$$\omega^2 - \omega_{c0}^2 - \frac{1}{3}\omega_{p0}^2 = \frac{2ak^2\omega_{p0}^2}{3\kappa} e^{2i(\kappa d - \varphi_1 - \varphi_2 - \pi/4)}, \tag{41}$$

where $\tan\varphi_1 = \alpha\kappa/k^2$ and $\tan\varphi_2 = \alpha/\kappa$. We deduce

$$\kappa d = \varphi_1 + \varphi_2 + \pi/4 + n\pi/2 \tag{42}$$

and

$$\omega^2 = B_0^2 + (-1)^n \frac{2ak^2\omega_{p0}^2}{3\kappa} \ , \tag{43}$$

where $B_0^2 = \omega_{c0}^2 + \frac{1}{3}\omega_{p0}^2$ and n is any integer. The solutions of equation (42) can be denoted by κ_n, $\kappa_n > 0$. The factor k^2/κ entering equation (43) can be written as $k^2/\kappa = (\lambda^2 - \kappa^2)/\kappa$. Within dipole approximation the condition $B_0 a/c \ll 1$ and κ close to λ should be fulfilled. Consequently, in this region we aproximate the factor $k^2/\kappa = (\lambda^2 - \kappa^2)/\kappa$ by $-2(\kappa - \lambda)$, for $\lambda - \kappa_1 < \kappa < \lambda + \kappa_2$, where $\Delta\kappa = \kappa_2 - \kappa_1$ is of the order of λ. The solution of equation (43) can then be written as

$$\omega_n \simeq B_0\sqrt{1 \pm C\kappa_n'} \ , \tag{44}$$

where $C = 4a\omega_{p0}^2/3B_0^2$, $\kappa_n' = \kappa_n - \lambda$ and $C\kappa_n' \ll 1$. In equation (44) we may represent $\Delta\kappa_n'$ as $\Delta\kappa_n' = \beta B_0/c$, where β is a factor of the order of the unity. An estimation for this factor can be obtained by comparing the first- and second-order derivatives of the function $\lambda^2/\kappa - \kappa$ for $\kappa = \lambda$. We get $\beta \simeq 2$. Under these circumstances, we can see also from equation (44) that α defined above remains a real quantity. We estimate the change produced in the energy of the electromagnetic field by the sphere-half-space coupling by the Euler-MacLaurin formula[48]

$$\Delta E = \sum_{m=1} \frac{(-1)^m B_m (\pi/d)^{2m-1}}{(2m)!} \left[f^{(2m-1)}(\kappa_1) - f^{(2m-1)}(\kappa_2) \right] \quad , \tag{45}$$

where B_m are Bernoulli's numbers and $f(\kappa) = \hbar B_0 (1 \pm C\kappa)^{1/2}$ ($\hbar$ being the Planck's constant). The leading contribution comes from the first-order derivative $f^{(1)}(\kappa) = \pm(1/2)\hbar B_0 C (1 \pm C\kappa)^{-1/2}$. The difference $\Delta f^{(1)} = f^{(1)}(\kappa_1) - f^{(1)}(\kappa_2)$ can be approximated by $\Delta f^{(1)} \simeq -f^{(2)}(0)\Delta\kappa = (1/4c)\beta\hbar B_0^2 C^2$, so we get the energy

$$\Delta E \simeq -\frac{4\pi}{9}\beta B_1 \frac{\hbar a^2 \omega_{p0}^4}{c B_0^2 d} = -\frac{2\pi}{27}\beta \frac{\hbar a^2 \omega_{p0}^4}{c B_0^2 d} \tag{46}$$

and the force (the leading term)

$$F = -\frac{2\pi}{27}\beta \frac{\hbar a^2 \omega_{p0}^4}{c B_0^2 d^2} \quad . \tag{47}$$

It is easy to see from equation (45) that higher-order corrections to this result are of the form $1/d^4$, $1/d^6$, etc (all attractive). Similar corrections are obtained if higher-order contributions are included beyond the dipole approximation. They result in higher powers of the k wavevector in equation (43) and, by equation (45), in higher-powers of the $1/d$. Under these circumstances, and bearing in mind that we only estimate the change in the energy due to the coupling, we may give up the restrictive condition $B_0 a/c \ll 1$. Similar results (to some extent) have been reported recently in Refs. [35, 36].

We can see that the force given by equation (47) has a different character than the Casimir force acting between an atom and a half-space (which goes like $1/d^5$). The characteristic d^2-dependence suggests a Coulomb force acting between a small particle (sphere) and its image in the half-space. From the estimation of the factor k^2/κ made above, we can see that this Casimir force involves the surface plasmon-polaritons modes in both bodies, damped along the z-axis and either propagating or damped along the in-plane directions in the half-space (parallel to the surface). These latter modes are reminiscent of the fluctuating modes.

In addition, from the standpoint of a quantum-mechanical treatment, we may view a macroscopic sphere as an infinite medium. Then, the relationship $d^{-n+3} - d^{-n}$ for $n-3 = 2$ gives $n = 5$, which is indeed the exponent of the quantum Casimir force acting between a quantum particle and a half-space.

van der Waals-London force

For shorter distances d the interaction becomes non-retarded, and we can take the limit $\lambda \to 0$ in the dispersion equation (40). We get

$$(\omega^2 - \omega_c^2 - \frac{1}{2}\omega_p^2)(\omega^2 - \omega_{c0}^2 - \frac{1}{3}\omega_{p0}^2) = \frac{1}{3}\omega_p^2 \omega_{p0}^2 a k e^{-2kd} \quad , \tag{48}$$

where we can recognize the frequency $\omega_p/\sqrt{2}$ of the well-known surface plasmons.[49] We may consider the *rhs* of equation (31) as a small perturbation ($ak \ll 1$). Introducing the notation $A^2 = \omega_c^2 + \omega_p^2/2$ (and $B_0^2 = \omega_{c0}^2 + \omega_{p0}^2/3$) the solutions of this equation can be written as

$$\begin{aligned} \omega_1 &\simeq A + \tfrac{\omega_p^2 \omega_{p0}^2}{6A(A^2 - B_0^2)} a k e^{-2kd} \ , \\ \omega_2 &\simeq B_0 - \tfrac{\omega_p^2 \omega_{p0}^2}{6B(A^2 - B_0^2)} a k e^{-2kd} \ . \end{aligned} \tag{49}$$

Since we are interested in the corrections brought about by the coupling to the total energy, we can leave aside, in fact, the restrictive conditions $A\omega/c$, $B_0\omega/c \ll 1$. The occurrence of such conditions is a feature of the classical approach, in contrast with the quantum-mechanical approach, where we have a direct access to the perturbation energy (see, for instance, Ref. [13]). The change ΔE brought by the interaction in the zero-point energy of the electromagnetic field (per unit area) is given by

$$\Delta E = -\frac{\hbar a}{96\pi}\frac{\omega_p^2\omega_{p0}^2}{AB_0(A+B_0)}\cdot\frac{1}{d^3} \ , \tag{50}$$

and the corresponding force (leading contribution) acquires the form

$$F = -\frac{\hbar a}{32\pi}\frac{\omega_p^2\omega_{p0}^2}{AB_0(A+B_0)}\cdot\frac{1}{d^4} \ . \tag{51}$$

This is the well-known, classical van der Waals-London force, acting between a quantum particle and a half-space (as well as between a point-like body and a half-space[38]). It corresponds to the $1/d^7$-law of interaction between two quantum particles (van der Waals-London).

There are higher-order corrections to these leading contributions going like $1/d^5$, $-1/d^6$, etc (with alternate signs). However, this behaviour is limited by $d \gg a$. There is a crossover from the retarded, Casimir force given by equation (47) and the non-retarded, van der Waals-London force given by equation (51). The crossover distance d is obtained by equating the two forces (with the cross-sectional area πa^2 of the sphere). It is of the order $d \sim \sqrt{ac/\omega_p}$, where ω_p is a representative frequency of the order of the plasma frequencies of the two bodies. We notice that $d \sim a\sqrt{c/a\omega_p} \gg a$ for $\omega_p a/c \ll 1$.

It is worth noting the great difference between the van der Waals-London and Casimir forces. The former implies the delocalized energy of the electromagnetic field acting upon the cross-sectional area, while the latter is associated with the electromagnetic energy, carried out by the radiation, localized bewteen the sphere and its image in the half-space. Out of all the electromagnetic frequencies, the subset of eigenfrequencies for the non-retarded interaction (labelled by the in-plane wavevector $\mathbf{k}$) is considerably larger than the corresponding subset of eigenfrequencies of the retarded interaction, labelled only by the one-dimensional set κ_n.

Two spheres

We consider two spheres, in the same conditions as above, one, with radius a and charge density n_0q placed at the origin and another, with radius b and charge density nq, placed at $z = d$. Their equation of motion has the same form as equation (32) (with parameters γ and, respectively, γ'). Making use of equation (37), we get the field created by the sphere placed at the origin

$$E_z = i\lambda A_z - \frac{\partial\Phi}{\partial z} = i\lambda A_z - \frac{1}{i\lambda}\frac{\partial^2 A_z}{\partial z^2} = \frac{ik^2}{\lambda}A_z \ , \tag{52}$$

where $A_z = (8\pi^2\lambda a^4/3\kappa)\gamma e^{i\kappa|z|}$ (equation (37)). Similarly, the field created by the sphere placed at the distance d can be obtained from this equation by changing z into $z-d$. We get two coupled equations of motion

$$\begin{aligned}(\omega^2-\omega_{c0}^2-\tfrac{1}{3}\omega_{p0}^2)\pi\gamma a^3 &= -\tfrac{2\pi i}{3}\omega_p^2\tfrac{b^4k^2}{\kappa}\gamma' e^{i\kappa d} \ ,\\ (\omega^2-\omega_c^2-\tfrac{1}{3}\omega_p^2)\pi\gamma' b^3 &= -\tfrac{2\pi i}{3}\omega_{p0}^2\tfrac{a^4k^2}{\kappa}\gamma e^{i\kappa d} \ ,\end{aligned} \tag{53}$$

which lead to the dispersion equation

$$(\omega^2-\omega_c^2-\frac{1}{3}\omega_p^2)(\omega^2-\omega_{c0}^2-\frac{1}{3}\omega_{p0}^2) = -\frac{4}{9}\omega_p^2\omega_{p0}^2 ab\frac{k^4}{\kappa^2}e^{2i\kappa d} \ . \tag{54}$$

The treatment of this equation is similar with the one given above for the couple sphere-half-space. We introduce the notation $B^2 = \omega_c^2 + \omega_p^2/3$ (and $B_0^2 = \omega_{c0}^2 + \omega_{p0}^2/3$), and see that the solution of equation (54) depends on whether the substances of the spheres are identical ($B = B_0$) or distinct ($B \neq B_0$). For two spheres consisting of identical substance, $B = B_0$, the results are the same as those corresponding to one sphere coupled to the half-space, except for the constant C in equations (46) and (47), which is replaced by $C = 4\omega_p^2\sqrt{ab}/3B^2$. With the image-force interpretation, this may not be an unexpected result. For two distinct substances ($B \neq B_0$), the leading contribution comes from the third-order derivative in equation (45), since the contribution arising from the first-order derivative is vanishing, as a consequence of the quadratic dependence of $k^4/\kappa^2 \simeq 4(\kappa - \lambda)^2$ in equation (54). The final result for the energy can be written as

$$\Delta E = -\frac{32\hbar}{1215c}\left(\frac{\pi}{d}\right)^3 \frac{\omega_p^4\omega_{p0}^4(B^2 + B_0^2)a^2b^2}{B^2B_0^2(B^2 - B_0^2)^2} \ . \tag{55}$$

We can see that the force goes like $1/d^4$. Comparing with the quantum-mechanical calculations, we can view the two spheres as two infinite media, and use the relationship $d^{-n+4} - d^{-n}$; hence, $n = 8$, which is indeed the exponent in the Casimir force acting acting beween two quantum particles. Similar results are discussed recently in Refs. [35, 36]

In the non-retarded limit ($\kappa = -ik$) equation (54) becomes

$$(\omega^2 - \omega_c^2 - \frac{1}{3}\omega_p^2)(\omega^2 - \omega_{c0}^2 - \frac{1}{3}\omega_{p0}^2) = \frac{4}{9}\omega_p^2\omega_{p0}^2 abk^2e^{-2kd} \ , \tag{56}$$

and the force (per unit area) is given by

$$F = -\frac{\hbar ab}{12\pi}\frac{\omega_p^2\omega_{p0}^2}{BB_0(B + B_0)} \cdot \frac{1}{d^5} \tag{57}$$

for distinct substances, and by

$$F = -\frac{\hbar ab\omega_p^4}{24\pi B^3} \cdot \frac{1}{d^5} \tag{58}$$

for identical substance. This force has no quantum-mechanical analog.

Point-like bodies

The displacement field for a point-like body placed at $\mathbf{R}_0$ can be taken as $a^3\mathbf{u}\delta(\mathbf{R} - \mathbf{R}_0)$, where a is the "radius" of the body and $\mathbf{u}$ is a constant vector (depending only on the time). The charge and current densities (temporal Fourier transforms) are given by

$$\rho = -a^3(\mathbf{u}grad)\delta(\mathbf{R} - \mathbf{R}_0) \ , \ \mathbf{j} = -ia^3\omega\mathbf{u}\delta(\mathbf{R} - \mathbf{R}_0) \ , \tag{59}$$

where the factor nq is left aside. The electromagnetic potentials given by equations (4) can be computed straightforwardly. They are given by

$$\Phi = -a^3(\mathbf{u}grad)F \ , \ \mathbf{A} = -ia^3\lambda\mathbf{u}F \ , \tag{60}$$

where

$$F = \frac{e^{i\lambda|\mathbf{R}-\mathbf{R}_0|}}{|\mathbf{R} - \mathbf{R}_0|} \ . \tag{61}$$

We use the Fourier transform $F(k, z) = (2\pi i/\kappa)e^{i\kappa|z-z_0|}$ given by equation (8) for the function F, and introduce the in-plane (transverse) components $u_{1,2}$, together with the z-component u_z, for the displacement. The field is obtained by $\mathbf{E} = i\lambda\mathbf{A} - grad\Phi$ from the above equations.

For $\mathbf{R}_0 = 0$ we get immediately the field

$$E_1 = a^3\kappa(\kappa u_1 - ku_z)F(k,z) \ , \ E_2 = a^3\lambda^2 u_2 F(k,z) \ ,$$
$$E_z = -a^3 k(\kappa u_1 - ku_z)F(k,z) \tag{62}$$

for $z > 0$ and $F(k,z) = (2\pi i/\kappa)e^{i\kappa z}$. Similarly, for the point-like body of radius b placed at $\mathbf{R}_0 = (0,\, 0,\, d)$, with the displacement denoted by $\mathbf{v}$, we get the field

$$E_1 = b^3\kappa(\kappa v_1 + kv_z)F(k,z) \ , \ E_2 = a^3\lambda^2 v_2 F(k,z) \ ,$$
$$E_z = b^3 k(\kappa v_1 + kv_z)F(k,z) \tag{63}$$

for $z < d$ and $F(k,z) = (2\pi i/\kappa)e^{i\kappa(d-z)}$.

We write the equations of motion (1) for two point-like bodies, making use of the combinations $\kappa u_1 - ku_z$ and $\kappa v_1 + kv_z$. We get two dispersion equations

$$(\omega^2 - \omega_{c0}^2)(\omega^2 - \omega_c^2) = -\tfrac{1}{4}\omega_p^2\omega_{p0}^2 ab(k^2/\kappa - \kappa)^2 e^{2i\kappa d} \ ,$$
$$(\omega^2 - \omega_{c0}^2)(\omega^2 - \omega_c^2) = -\tfrac{1}{4}\omega_p^2\omega_{p0}^2 ab(k^2/\kappa + \kappa)^2 e^{2i\kappa d} \tag{64}$$

(the latter equation comes from the coordinates u_2, v_2). These two equations are not compatible with one another. We choose vanishing transverse components, $u_2 = v_2 = 0$, and we are left with the first equation (64), which can be treated similarly as equation (54). The analysis is similar for the second dispersion equation (64), corresponding to a vanishing displacement component along the z-axis.

First, we consider an identical substance, $\omega_{c0} = \omega_c$. The solutions of the first equation (64) are given by $\kappa_n d = n\pi/2$ and

$$\omega^2 = \omega_c^2 \pm \frac{1}{2}\omega_p^2\sqrt{ab}(k^2/\kappa - \kappa) \ . \tag{65}$$

The factor $k^2/\kappa - \kappa = \lambda^2/\kappa - 2\kappa$ can be expanded in powers of $\kappa - \kappa_0$, where, in the limit $a,\, b \to 0$ the κ_0-term is immaterial. Since in the limit $a,\, b \to 0$ the exact result given by equation (65) amounts to the dipole approximation, we may take κ_0 as the value of κ which nullifies the factor $k^2/\kappa - \kappa$: $\kappa_0 = \lambda/\sqrt{2}$. Equation (65) can then be cast in the form

$$\omega_n = \omega_c\sqrt{1 \pm C\kappa_n} \ , \tag{66}$$

where κ_n varies around zero within the interval $\Delta\kappa = \beta\lambda = \beta\omega_c$, β being a numerical factor of the order of the unity ($\beta \simeq 1/2$). We apply the Mac-Laurin summation given by equation (45), the final result being

$$\Delta E = -\frac{\pi}{6}\beta\frac{\hbar ab\omega_p^4}{c\omega_c^2 d} \ . \tag{67}$$

We can see that the force goes like $1/d^2$, the situation being similar with the interaction between two-identical spheres (or a sphere and a half-space). For distinct substances, it is the third-order derivative which contributes to the Euler-Mac-Laurin summation (due to the quadratic factor $(k^2/\kappa - \kappa)^2$), the situation is similar with two interacting spheres, and the Casimir force goes like $1/d^4$. In the non-retarded limit ($\kappa = ik$) the first equation (64) has the same form as equation (56) for two spheres, and the force goes like $1/d^5$. The coupling between a point-like body and a sphere is also similar with the coupling between two spheres. Different other situations may appear, related to these equations, depending on whether one body is "conducting", or both bodies are "conducting". In this case, $\omega_{c0} = 0$, for instance, or both characteristic frequencies are vanishing, $\omega_c = \omega_{c0} = 0$. These situations are treated similarly, by the same method described here. Such special situations arise from the fact that we do not allow a dynamics for the (internal) polarization of the localized point-like bodies, which is rather a special, unrealistic assumption. This is why we do not follow further such cases here. Similar situations may appear also for spherical shells of a vanishing thickness. The field generated by a spherical shell can be calculated by the same method as the one presented here for the sphere, and a large variety of coupling involving spherical shells can be treated. It is worth noting the great variety of situations which can be investigated by the the classical interaction between the electromagnetic field and macroscopic bodies.

The coupling between a point-like body and a half-space is similar with the coupling between a sphere and a half-space. We use the coupled equations (18) for the half-space with the (external) field given by equations (62) (the field generated by the sphere) and the equations of motion (1) for the point-like body with the field given by equations (19) (the field of the half-space). For vanishing transverse components of the displacement the dispersion equation is given by

$$\omega^2 - \omega_{c0}^2 = \frac{i}{2}\omega_{p0}^2 a \frac{\kappa^2 - k^2}{\kappa} \frac{\kappa\kappa' - k^2}{\kappa\kappa' + k^2} \frac{\kappa' - \kappa}{\kappa' + \kappa} e^{2i\kappa d} \ . \tag{68}$$

This equation is analogous with the dispersion equations (40) and (41) for the couple sphere-half-space, except (beside numerical coefficients) for the factor $k^2/\kappa - \kappa$, which appears in the place of the factor k^2/κ in equation (41). The treatment of the equation (68) is analogous with the treatment done for the equations (40) and (41), resulting a Casimir force $\sim 1/d^2$ and a van der Waals-London force $\sim 1/d^4$. A special case is the "conducting" point-like body, for which $\omega_{c0} = 0$. In this, rather unrealistic, case, it is easy to see that the Casimir force is vanishing, while the van der Waals-London force is repulsive.

Spherical shells

By analogy with the sphere, the displacement field for a spherical shell of radius a and thickness ε, placed at the origin, can be written as

$$\mathbf{u} = \varepsilon a \gamma (0,\, 0,\, 1)\delta(R - a) \ . \tag{69}$$

The electromagnetic field can be computed from the potentials in the same manner as for the sphere. Within our approximation, the field inside the sphere is vanishing. The external field is given by

$$E_1 = -\frac{k}{\lambda}\frac{\partial A_z}{\partial z} \ , \ E_z = i\lambda A_z + \frac{i}{\lambda}\frac{\partial^2 A_z}{\partial z^2} \ , \tag{70}$$

where

$$A_z = 4\pi\lambda^2 a^3 \varepsilon\gamma h_0(\lambda R) \tag{71}$$

(h_0 being the Hankel function o fthe zeroth order). For the Fourier transforms we get

$$E_1 = -8\pi^2 i a^3 \varepsilon\gamma k e^{i\kappa z} \ , \ E_z = 8\pi^2 i a^3 \varepsilon\gamma \frac{k^2}{\kappa} e^{i\kappa z} \ . \tag{72}$$

This field can be used for coupling the spherical shell with any other body described here, by using the equation of motion (1). The dispersion equations are very similar with the equations for a sphere, except for the contribution of the (internal) polarization (which we do not allow for a spherical shell). The results do not depend on the thickness ε.

A "special" case

The field of a point-like body can be computed straightforwardly in the direct space, making use of the potentials given by equations (59) (without resorting to the Fourier tranforms). We may give up the sharpness of the surface of the macroscopic bodies, as expressed by the function δ in the displacement field given by $a^3\mathbf{u}\delta(\mathbf{R}-\mathbf{R}_0)$. Such a special case may sometimes be viewed as a "classical" representation for quantum particles. The equations of motion (1) can then be written in the direct space (not by using Fourier transforms). We do so for two point-like particles placed at $\mathbf{R}_0=0$ and $\mathbf{R}_0=(0,\,0,\,d)$, as before. We choose vanishing transverse components of the displacement, and we are left with the dispersion equation

$$(\omega^2-\omega_{c0}^2)(\omega^2-\omega_c^2)=\frac{a^3b^3}{4\pi^2d^3}\omega_{p0}^2\omega_p^2(1-i\lambda d)^2e^{2i\lambda d} \tag{73}$$

for the displacement component along the z-axis. In the non-retarded case ($\lambda=0$), it is easy to see that this equation leads to the attractive $1/d^7$-van der Waals-London force (for both ω_c, $\omega_{c0}\neq 0$) (if $\omega_c=0$, or $\omega_{c0}=0$, the force is repulsive; for both $\omega_c=\omega_{c0}=0$, the force is repulsive and goes like $1/d^{7/2}$). A similar conclusion is reached for the transverse displacement components (vanishing displacement along the z-direction).

In the retarded regime, the solutions of the equation (73) are given by

$$\lambda d-\varphi=n\pi/2\ , \tag{74}$$

where $\tan\varphi=\lambda d$, and

$$(\omega^2-\omega_{c0}^2)(\omega^2-\omega_c^2)=\pm\frac{a^3b^3}{4\pi^2d^3}\omega_{p0}^2\omega_p^2(1+\lambda^2d^2)^2 \tag{75}$$

(for the displacement along the z-axis). The analysis of the solutions of equation (75) depends on the parameters ω_c, ω_{c0}. We consider here the most interesting case of identical particle, $\omega_c=\omega_{c0}$. Equation (75) can be solved easily, and we can see that the force is vanishing. A similar conclusion holds for the dispersion equation of the transverse components of the displacement. This proves the inadequacy of such a model for quantum particles (beside its incorrect use - in the direct space - for macroscopic bodies).

Concluding remarks

The well-known van der Waals-London and Casimir forces are derived by quantum mechanical calculations (in the non-retarded and, respectively, retarded regime). The origin of these forces resides in the polarization of the material bodies. The macroscopic bodies exhibit their own polarization characteristics, in comparison with the quantum particles. A method has been developed here for treating the classical interaction between the electromagnetic field and the polarizable matter, in order to derive the macroscopic counterpart of the van der Waals-London and Casimir forces. The method has been applied here to the couples sphere-half-space, two spheres, two point-like bodies, a point-like body and a half-space or a sphere (in general, a point-like body behaves, in this respect, very much alike a sphere). The coupling of two half-spaces has been discussed in Ref. [37].

The method is based on representing the polarization by a displacement field $\mathbf{u}$ of the mobile charges, which obeys the classical (Newton) equation of motion. The well-known Lorentz-Drude (plasma) model for polarizable matter has been used in this respect. The coupled equations of motion are solved for the electromagnetic eigenmodes, both in the retarded and non-retarded regime, and the coupling energy is estimated as the correction brought by interaction to the zero-point energy of the electromagnetic field (vacuum fluctuations; thermal corrections can be included). The dipole approximation have been used for the electromagnetic field of a sphere, in order to estimate the leading contributions to the interaction.

For a sphere coupled to a half-space the Casimir force goes like $1/d^2$ (at large distance) and the van der Waals-London force goes like $1/d^4$, where d is the separation distance between the two bodies. The latter is the same as the one obtained by quantum-mechanical calculations, the former agrees with the quantum-mechanical calculations providing that the sphere is viewed as an infinite medium. For two interacting spheres the Casimir force goes like $1/d^2$ if the spheres are made of the same substance, and it goes like $1/d^4$ if the substances are diferent. The relationship with the quantum-mechanical calculations is the same as for a sphere and a half-space. The van der Waals-London force for two interacting spheres goes like $1/d^5$; it has no quantum-mechanical analog. The point-like bodies behave similarly with the spheres.

Acknowledgments. The author is indebted to the members of the Laboratory of Theoretical Physics and Condensed Matter in the Institute for Physics and Nuclear Engineering, Magurele-Bucharest, and the Seminar of the Institute for Atomic Physics, Magurele-Bucharest, for many useful discussions.

References

[1] H. B. G. Casimir and D. Polder, "The influence of retardation on the London-van der Waals forces," Phys. Rev. **73** 360-372 (1948).

[2] H. Casimir, "On the atttraction between two perfectly conducting plates," Proc. Kon. Ned. Ak. Wet. **51** 793 (1948).

[3] R. Eisenschitz and F. London, "Uber das Verhaltnis der van der Waalsschen Krafte zu den homoopolaren Bindungskraften", Z. Physik **60** 491-527 (1930).

[4] F. London, "Zur Theorie und Systematik der Molekularkrafte", Z. Physik **63** 245-279 (1930).

[5] F. London, "The general theory of molecular forces", Trans. Faraday Soc. **33** 8-26 (1937).

[6] E. Lifshitz, ZhETF **29** 94-105 (1956) (Sov. Phys. JETP **2** 73-83 (1956)).

[7] I. E. Dzyaloshinskii, E. M. Lifshitz and L. P. Pitaevskii, "The general theory of van der Waals forces," Adv. Phys. **10** 165-209 (1961).

[8] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 5 (*Statistical Physics*), part. 2, Butterworth-Heinemann, Oxford (2003).

[9] J. Schwinger, "Casimir effect in source theory", Lett. Math. Phys. **1** 43-47 (1975).

[10] J. Schwinger, L. L. deRaad, Jr., and K. A. Milton, "Casimir effect in dielectrics", Ann. Phys. **115** 1-23 (1978).

[11] N. G. Van Kampen, B. R. A. Nijboer and K. Schram, "On the macroscopic theory of van der Waals forces," Phys. Lett. **A26** 307-308 (1968).

[12] T. H. Boyer, "Quantum electromagnetic zero-point energy of a conducting spherical shell and the Casimir model for a charged particle", Phys. Rev. **174** 1764-1776 (1968).

[13] T. H. Boyer, "Recalculations of long-range van der Waals potentials", Phys. Rev. **180** 19-24 (1969).

[14] E. Gerlach, "Equivalence of van der Waals forces between solids and the surface-plasmon interaction," Phys. Rev. **B4** 393-396 (1971).

[15] K. Schram, "On the macroscopic theory of retarded van der Waals forces," Phys. Lett. **A43** 282-284 (1973).

[16] J. Heinrichs, "Theory of van der Waals interaction between metal surfaces," Phys. Rev. **B11** 3625-3636 (1975).

[17] P. W. Milloni, *The Quantum Vacuum* (Academic Press, San Diego, 1994).

[18] V. M. Mostepanenko and N. N. Trunov, *The Casimir Effect and its Applications* (Clarendon, Oxford, 1997).

[19] S. K. Lamoreaux, "Demonstration of the Casimir force in the 0.6 to $6\mu m$ range," Phys. Rev. Lett. **78** 5-8 (1997).

[20] L. H. Ford, "Casimir sphere between a dielectric sphere and a wall: a model for amplification of vacuum fluctuations, Phys. Rev. **A58** 4279-4286 (1998).

[21] U. Mohideen and A. Roy, "Precision measurement of the Casimir force from 0.1 to $0.9\mu m$", Phys. Rev. Lett. **81** 4549-4552 (1998).

[22] S. K. Lamoreaux, "Calculation of the Casimir force between imperfectly conductiong plates," Phys. Rev. **A59** R3149-R3153 (1999).

[23] A. Lambrecht and S. Reynard, "Comment on "Demonstration of the Casimir force in the 0.6 to $6\mu m$ range"," Phys. Rev. Lett. **84** 5672-5672 (2000).

[24] A. Lambrecht and S. Reynaud, "Casimir force between metallic mirrors", Eur. Phys. J. **D8** 309-318 (2000).

[25] M. Bordag, U. Mohideen and V. M. Mostepanenko, "New developments in the Casimir effect," Phys. Reps. **353** 1-205 (2001).

[26] K. A. Milton, *The Casimir Effect* (World Scientific, Singapore, 2001).

[27] G. Bressi, G. Carugno, R. Onofrio and G. Ruoso, "Measurement of the Casimir force between parallel metallic surfaces", Phys. Rev. Lett. **88** 041804 (2002) (1-4).

[28] C. Genet, A. Lambrecht and S. Reynaud, "Casimir force and the quantum theory of lossy optical cavities," Phys. Rev. **A67** 043811 (1-18) (2003).

[29] K. A. Milton, "The Casimir effect: recent controversies and progress", J. Phys. A: Math. Gen. **37** R209-R277 (2004).

[30] F. Chen, U. Mohideen, G. L. Klimchitskaya and V. M. Mostepanenko, "Investigation of the Casimir force between metal and semiconductor test bodies," Phys. Rev. **A72** 020101(R1-4) (2005).

[31] S. K. Lamoreaux, "The Casimir force: background, experiments and applications," Reps. Progr. Phys. **65** 201-236 (2005).

[32] F. Intravaia, "Effet Casimir et interaction entre plasmons de surface," These de Doctorat de l'Universite Paris VI, 1-177, June 2005.

[33] J. M. Obrecht, R. J. Wild, M. Antezza, L. P. Pitaevskii, S. Stringari and E. A. Cornell, "Measurement of the temperature dependence of the Casimir-Polder force," Phys. Rev. Lett. **98** 063201 (1-4) (2007).

[34] F. Intravaia, C. Henkel and A. Lambrecht, "Role of surface plasmons in the Casimir effect," Phys. Rev. **A76** 033820 (1-11) (2007).

[35] T. Emig, N. Graham, R. L. Jaffe and M. Kardar, "Casimir forces between arbitrary compact objects", Phys. Rev. Lett. **99** 170403 (2007) (1-4).

[36] M. Kruger, T. Emig, G. Bimonte and M. Kardar, "Non-equilibrium Casimir forces:: spheres and sphere-plate", Eur. Phys. Lett. **95** 21002 (2011) (1-6).

[37] M. Apostol and G. Vaman, "Electromagnetic eigenmodes in matter. van der Waals-London and Casimir forces", Pogr. Electrom. Res. PIER **B19** 115-131 (2010).

[38] M. Apostol and G. Vaman, "Attraction force between a polarizable point-like particle and a semi-infinite solid", Roum. J. Phys. **55** 764-771 (2010).

[39] P. Drude, "Zur Elektronentheorie der Metalle", Ann. Phys. **306** 566-613 (1900).

[40] P. Drude, "Zur Elektronentheorie der Metalle, 2. Teile. Galvanomagnetische und thermomagnetische Effecte, **308** 369-402 (1900).

[41] H. A. Lorentz, *The Theory of Electrons*, Teubner, Leipzig (1916).

[42] N. W. Ashcroft and N. D. Mermin, *Solid State Physics*, Saunders College,Philadelphia (1976).

[43] G. D. Mahan, *Many-Particle Physics*, Plenum, NY (1980).

[44] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, eds. A. Jeffrey and D. Zwillinger, 6th edition, Academic Press, San Diego (2000), pp. 714-715, 6.677.1,2.

[45] M. Born and E. Wolf, *Principles of Optics*, Pergamon, London (1959).

[46] M. Apostol and G. Vaman, "Plasmons and difraction of an electromagnetic plane wave by a metallic sphere", Progr. Electrom. Res. PIER **98** 97-118 (2009).

[47] G. Mie, "Beitrage zur Optik truber Medien, speziell kolloidaler Metallosungen", Ann. Physik **25** 377-445 (1908).

[48] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Mathematical Library, Cambridge (2004), p. 128.

[49] M. Apostol and G. Vaman, "Electromagnetic field interacting with a semi-infinite plasma", J. Opt. Soc. Am. **A26** 1747-1753 (2009).