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The electromagnetic field of linear, or circular, antennas. The dipole approximation

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Abstract

The electromagnetic field produced by a linear antenna of finite length are derived within the dipole approximation. Some technical points are included, regarding mathematical arcana in solving the Helmholtz equation. Similar results are given for a circular antenna. The dipole approximation is generalized for a body of arbitrary shape. Both radiation fields and near-field approximation are discussed. The electromagnetic field inside small bodies is also estimated within the dipole approximation, and the well-known iridiscence caused by the polarization resonance is derived.

Linear antenna. We consider a linear antenna of length l placed between $z=0$ and $z=l$, with a small thickness $a \ll l$. The displacement field of the mobile charges q, with densiy n is given by

$$\mathbf{u}(\mathbf{R},t)=a^2\delta(\mathbf{r})v(z,t)\theta(z)\theta(l-z)\mathbf{e}_z \ , \tag{1}$$

where $\mathbf{R}=(\mathbf{r},z)$, δ is the delta function,θ is the step function, $v(z,t)$ is a function of the coordinate z and the time t and $\mathbf{e}_z$ is the unit vector along the z-direction. Thsi displacement field generates a charge density

$$\rho=-nq div\mathbf{u}=-nqa^2\delta(\mathbf{r})\left[\frac{\partial v}{\partial z}\theta(z)\theta(l-z)+v\delta(z)-v\delta(z-l)\right] \tag{2}$$

and a current density

$$\mathbf{j}=nqa^2\delta(\mathbf{r})\frac{\partial v}{\partial t}\theta(z)\theta(l-z)\mathbf{e}_z \ , \tag{3}$$

both satisfying the continuity equation $\partial\rho/\partial t+div\mathbf{j}=0$. We use the temporal Fourier transform with frequency denoted by ω and the well-known Kirchhoff's retarded electromagnetic potentials

$$\begin{array}{c}\mathbf{A}(\mathbf{R},t)=\frac{1}{c}\int d\mathbf{R}'\frac{\mathbf{j}(\mathbf{R}',t-|\mathbf{R}-\mathbf{R}'|/c)}{|\mathbf{R}-\mathbf{R}'|} \ , \\ \Phi(\mathbf{R},t)=\frac{1}{c}\int d\mathbf{R}'\frac{\rho(\mathbf{R}',t-|\mathbf{R}-\mathbf{R}'|/c)}{|\mathbf{R}-\mathbf{R}'|} \ ,\end{array} \tag{4}$$

subjected to the Lorenz gauge $div\mathbf{A}+(1/c)\partial\Phi/\partial t=0$ (with c the light velocity). We omit usually the argument ω in the Fourier transforms. The vector potential has only the z-component, $\mathbf{A}=(0,\,0,\,A)$. We get

$$A=-inqa^2\lambda\int_0^l dz'\frac{v(z')}{\sqrt{r^2+(z-z')^2}}e^{i\lambda\sqrt{r^2+(z-z')^2}} \tag{5}$$

and

$$\Phi = nqa^2 \int_0^l dz' v(z') \frac{\partial}{\partial z'} \frac{1}{\sqrt{r^2+(z-z')^2}} e^{i\lambda\sqrt{r^2+(z-z')^2}} \ . \tag{6}$$

For $l/R \ll 1$, and introducing the mean value

$$\int_0^l dz v(z) = l\overline{v} \tag{7}$$

we get immediately

$$A = -ip\lambda \frac{e^{i\lambda R}}{R} \ , \ \Phi = -p\left(\frac{i\lambda z}{R} - \frac{z}{R^2}\right)\frac{e^{i\lambda R}}{R} \ , \tag{8}$$

where $p = nqa^2 l\overline{v} = Nq\overline{v}$ is the dipole moment, N being the total number of mobile charges. This approximation is also called the dipole approximation ($l/R \ll 1$). We can check immediately the Lorenz gauge on this approximation. If $\overline{v} = 0$ we take further terms in the expansion of $\sqrt{r^2+(z-z')^2}$ up to the leading contribution.

From equations (8) we can get the electric field $\mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t - grad\Phi$ and the magnetic field $\mathbf{H} = curl\mathbf{A}$. The dipole approximation is valid for $l/R \ll 1$, irrespective of the relation between R, l and the wavelength $c/\omega = 1/\lambda$. We limit ourselves here to give the radiation fields in the wave-zone, *i.e.* for $\lambda R \gg 1$ (distances much longer than the wavelength):

$$\begin{gathered} E_z = p\lambda^2 \tfrac{e^{i\lambda R}}{R} \ , \ \mathbf{E}_\perp = -p\lambda^2 \tfrac{z\mathbf{r}}{R^3} e^{i\lambda R} \ , \\ H_\varphi = -p\lambda^2 \tfrac{r\sin\theta}{R^2} e^{i\lambda R} \ , \end{gathered} \tag{9}$$

where $E_\perp$ is the transverse electric field ($x = r\cos\varphi$, $y = r\sin\varphi$) and H_φ is written in spherical coordinates ($z = R\cos\theta$, $r = R\sin\theta$). We can see that these fields are (distorted) spherical waves, with a significant dependence on direction. The E_z-component of the electric field is purely a spherical wave, the transverse (radial) $\mathbf{E}_\perp$-component of the electric field is vanishing along the azimuthal and polar directions, the magnetic field is axial and everywhere perpendicular to the electric field, as for a plane wave.

We comment further on other possible techniques of solving this problem. Obviously, the problem concerns the solution of the wave equations for the electromagnetic potentials with sources ρ and $\mathbf{j}$ given by equations (2) and (3). It is convenient to deal first with the wave equation for the vector potential A, and derive thereafter the scalar potential from the Lorenz gauge. By a temporal Fourier transform the wave eqaution becomes the Helmholtz equation, which, leaving aside the irrelevant factors, can be written as

$$\Delta A + \lambda^2 A = \delta(\mathbf{r})w(z) \ , \tag{10}$$

where $w(z)$ is a function extending from $z = 0$ to $z = l$ (vanishing outside this region). The most direct approach to this equation would be to use the Green function for the equation

$$\Delta G + \lambda^2 G = \delta(\mathbf{r})\delta(z) = \delta(\mathbf{R}) \ , \tag{11}$$

which, as it is well known, is the spherical wave $-(1/4\pi)e^{i\lambda R}/R$ (for outgoing wave boundary conditions at infinity). Then, it is easy to see that the solution of equation (10) is given by convolutions with the Green function, of the Kirchhoff's solution-type given by equations (4).

However, we can choose to perform first a z-Fourier transform in equation (10), as a faster way to the solution. Here, there may appear a first difficulty. Let us take the one-dimensional Helmholtz equation

$$\frac{\partial^2 u}{\partial z^2}+\kappa^2 u=f(z) \ , \tag{12}$$

with common notations. Naively, we can write at once the "solution", by Fourier transforming the equation:

$$u(z)=\frac{1}{2\pi}\int dk\frac{f(k)}{\kappa^2-k^2}e^{ikz} \ ; \tag{13}$$

and see immediately that we need to specify the path of integration. We should go back to the equation for the Green function

$$\frac{\partial^2 G}{\partial z^2}+\kappa^2 G=\delta(z) \tag{14}$$

and see that solution is of the form $G\sim Ae^{\pm i\kappa z}$ for $z<0$ and $z>0$, with the "boundary condition" $G^{'}(\varepsilon)-G^{'}(-\varepsilon)=1$, $\varepsilon\rightarrow 0$. We can check immediately that it is given by

$$G(z)=\frac{1}{2i\kappa}e^{i\kappa|z|} \tag{15}$$

and its Fourier transform is

$$G(k)=\frac{1}{\kappa^2-k^2+i\kappa\mu} \tag{16}$$

for $\mu\rightarrow 0^+$. The "solution" (13) becomes

$$u(z)=\frac{1}{2\pi}\int dk\frac{f(k)}{\kappa^2-k^2+i\kappa\mu}e^{ikz} \ , \tag{17}$$

where, now, the poles are placed without ambiguity.

We turn now to equation (10), written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)+\frac{\partial^2 A}{\partial z^2}+\lambda^2 A=\delta(\mathbf{r})w(z) \ , \tag{18}$$

and take the z-Fourier transform (obviously, the angular part is not implied here in the laplacian) . According to the above discussion the precise imaginary part of the z-wavevector κ, which specifies the integration path, cannot be determined. Leaving it aside and denoting $k^2=\lambda^2-\kappa^2$, we get the Green-function equation

$$A^{''}+\frac{1}{r}A^{'}+k^2 A=\delta(\mathbf{r})w(\kappa) \ . \tag{19}$$

For $\mathbf{r}\neq 0$ the solution is given by the Bessel functions of zero-th order, say $CZ_0(z=kr)$, where C is a constant. It should satisfy the "boundary condition" $2\pi rC\frac{\partial Z_0}{\partial r}\rightarrow 1$ for $r\rightarrow 0$ (obtained by the Gauss's theorem for integral of div) . On the other hand, the Hankel function $H_0^{(1)}(z)$ goes like $(2i/\pi)\ln z$ for $z\rightarrow 0$ (and is outgoing wave for k in the upper half-plane), so we get finally

$$A(r,\kappa)=\frac{1}{4i}H_0(kr)w(\kappa)=\frac{1}{4i}H_0^{(1)}\left(\sqrt{\lambda^2-\kappa^2}r\right)w(\kappa) \ . \tag{20}$$

Unfortunately, the integration path is not specified, so it is difficult to get the inverse Fourier transform $A(r,z)$. This can be seen immediately for contributions arising from $\kappa\simeq\lambda$, where the Hankel function has a starting branch cut along the negative axis.

All what we can get is the Fourier transform of the Green function

$$\frac{1}{2\pi}\int d\kappa G(r,\kappa)e^{i\kappa z}=\frac{1}{8\pi i}\int d\kappa H_0^{(1)}(kr)e^{i\kappa z}=G(R)=-\frac{1}{4\pi}\frac{e^{i\lambda R}}{R} \ , \tag{21}$$

whence

$$H_0^{(1)}(kr)=\frac{1}{i\pi}\int dz\frac{e^{i\lambda R}}{R}e^{-i\kappa z} \ . \tag{22}$$

Indeed, with $z=r\sinh t$ and $\lambda=k\cosh\alpha$, $\kappa=k\sinh\alpha$, the integral in equation (22) becomes

$$H_0^{(1)}(kr)=\frac{1}{i\pi}\int dte^{ikr\cosh t} \ , \tag{23}$$

which is a known integral representation of the Hankel function.

It is worth computing $w(\kappa)$ for a function with a finite support, $0<z<l$. Let us consider $w(z)=we^{i\kappa_0 z}$, then

$$w(\kappa)=iw\frac{e^{i(\kappa_0-\kappa)l-\mu l}-1}{\kappa-\kappa_0-i\mu} \ . \tag{24}$$

We can use this Fourier transform in equation (20) in order to compute $A(r,z)$. It is easy to see that the contribution from the pole $\kappa_0+i\mu$ gives a function $A(r,z)$ with the same finite support $0<z<l$; had we know the full path of integration the correct inverse Fourier transform would be obtained. However, in the limit $l\rightarrow 0$ we get $w(\kappa)=wl$, and we get the correct result

$$A(r,z)=\frac{wl}{8\pi i}\int d\kappa H_0^{(1)}(kr)e^{i\kappa z}=-\frac{wl}{4\pi}\frac{e^{i\lambda R}}{R} \ , \tag{25}$$

as a result of equation (21).

Circular antenna. We consider a circular antenna of radius r_0 and thickness $a\ll r_0$, placed in the x, y-plane with the center at the origin. The displacement field of the mobile charges is given by

$$\mathbf{u}(\mathbf{R},t)=a^2\delta(z)\delta(r-r_0)v(\varphi,t)\mathbf{e}_\varphi \ , \tag{26}$$

where $\mathbf{R}=(\mathbf{r},z)$ and φ is the azimuthal angle. The charge and current densities are given by $\rho=-nqdiv\mathbf{u}$ and, respectively, $\mathbf{j}=nq\dot{\mathbf{u}}$, where n is the density of the mobile charges q. The Kirchhoff's potentials can be written immediately. For instance, the vector potential is given by

$$A_x=i\lambda nqa^2r_0\int d\varphi'\frac{v(\varphi')\sin\varphi'}{\sqrt{R^2-2rr_0\cos(\varphi-\varphi')+r_0^2}}e^{i\lambda\sqrt{R^2-2rr_0\cos(\varphi-\varphi')+r_0^2}} \tag{27}$$

and a similar expression for A_y (with the temporal Fourier transforms).

We limit ourselves to the dipole approximation $R\gg r_0$, and consider the most common case $v(\varphi)=v=const.$ We get straightforwardly $A_r=A_z=0$ and

$$A_\varphi=-\frac{1}{2}i\lambda nqa^2r_0^2v\left(\frac{r}{R^2}-\frac{i\lambda r}{R}\right)\frac{e^{i\lambda R}}{R}=-\frac{1}{2}i\lambda pr_0\left(\frac{r}{R^2}-\frac{i\lambda r}{R}\right)\frac{e^{i\lambda R}}{R} \ , \tag{28}$$

where $p=Nqv$ is the dipole moment, N being the total number of the mobile charges ($n=N/a^2r_0$). The radiation (wave-zone) field is given by

$$\begin{gathered}E_\varphi=-\tfrac{1}{2}i\lambda^3pr_0\tfrac{r}{R^2}e^{i\lambda R} \ , \\ H_r=\tfrac{1}{2}i\lambda^3pr_0\tfrac{rz}{R^3}e^{i\lambda R} \ , \ H_z=-\tfrac{1}{2}i\lambda^2pr_0\tfrac{1}{R^2}e^{i\lambda R} \ .\end{gathered} \tag{29}$$

We can see that these fields are dimished, in comparison with the fields of a linear antenna by a factor λr_0, and, as for a linear antenna, they are plane-wave fields. It is worth noting here that instead of v, which is directly associated with the dipole moment, there appears vr_0, as a consequence of the next-order term in the expansion in powers of r_0. Such terms are usually seen as quadrupole contribution, though they do not imply v^2. Properly assessed, such terms arise from quadratic combinations of the displacement and the dimensions of the body, and their origin is actually dipolar.

Generalized dipole approximation. Let us consider the temporal Fourier transform of the vector potential given by equations (4),

$$\mathbf{A} = \frac{1}{c}\int d\mathbf{R}'\frac{\mathbf{j}(\mathbf{R}')}{|\mathbf{R}-\mathbf{R}'|}e^{i\lambda|\mathbf{R}-\mathbf{R}'|} \ , \tag{30}$$

and assume the dipole approximation $R \gg R'$. The leading contribution to equation (30) is given by

$$\mathbf{A} = \frac{1}{c}\int d\mathbf{R}'\mathbf{j}(\mathbf{R}') \cdot \frac{e^{i\lambda R}}{R} \ . \tag{31}$$

We denote the average current density in equaton (31) by $v\mathbf{j}$, where v is the volume of the body:

$$\mathbf{A} = \frac{v}{c}\mathbf{j}\frac{e^{i\lambda R}}{R} \ . \tag{32}$$

If this average value is vanishing we take higher-order terms in the expansion of $e^{i\lambda|\mathbf{R}-\mathbf{R}'|}/\,|\mathbf{R}-\mathbf{R}'|$. From the continuity equation we get the charge density $\rho = (1/i\omega)div\mathbf{j}$, so that the scalar potential in equations (4) can be written as

$$\Phi = \frac{1}{i\omega}\int d\mathbf{R}'\frac{div\mathbf{j}}{|\mathbf{R}-\mathbf{R}'|}e^{i\lambda|\mathbf{R}-\mathbf{R}'|} \ , \tag{33}$$

or, within the same approximation,

$$\Phi = \frac{v}{i\omega}(\mathbf{j}grad)\frac{e^{i\lambda R}}{R} \ . \tag{34}$$

We can check the Lorenz gauge $div\mathbf{A} - i\lambda\Phi = 0$. Now it is easy to get the electric and the magnetic fields. In the wave-zone ($\lambda R \gg 1$) the radiation fields are given by

$$\mathbf{E} = \frac{iv\lambda}{c}\left[\mathbf{j} - \frac{(\mathbf{jR})\mathbf{R}}{R^2}\right]\frac{e^{i\lambda R}}{R} \ , \ \mathbf{H} = -\frac{iv\lambda}{c}(\mathbf{j}\times\mathbf{R})\frac{e^{i\lambda R}}{R^2} \ . \tag{35}$$

We can see that the fields are perpendicular to each other, as for a plane wave, and both are perpendicular to the direction $\mathbf{R}$ of the propagation of the wave. We can see also that only the transverse component of the current (perpendiular to $\mathbf{R}$) is effective.

The Poynting vector averaged over a large time T is given by

$$\mathbf{S} = \frac{c}{8\pi^2 T}\int d\omega \mathbf{E}(\omega)\times\mathbf{H}(-\omega) \ , \tag{36}$$

or, making use of equations (35),

$$\mathbf{S} = \frac{v^2}{8\pi^2 cT}\int d\omega\lambda^2\left(|\mathbf{j}|^2 - \frac{|\mathbf{jR}|^2}{R^2}\right)\frac{\mathbf{R}}{R^3} \ . \tag{37}$$

Suppose that $\mathbf{j}(t)$ is of the form $\mathbf{j}(t) = \mathbf{j}_0 e^{-\Omega t} + c.c.$, and $\mathbf{j}(\omega) = 2\pi \mathbf{j}_0 \delta(\omega - \Omega) + 2\pi \mathbf{j}_0 \delta(\omega + \Omega)$. Making use of $\delta^2(\omega - \Omega) = (T/2\pi)\delta(\omega - \Omega)$, we get the Poynting vector

$$\mathbf{S} = \frac{\Omega^2 v^2}{2\pi c^3}\left(|\mathbf{j}_0|^2 - \frac{|\mathbf{j}_0 \mathbf{R}|^2}{R^2}\right)\frac{\mathbf{R}}{R^3} \ . \tag{38}$$

Similarly, the density of the electromagnetic field is given by

$$W = \frac{v^2}{8\pi^2 c^2 T}\int d\omega \lambda^2 \left(|\mathbf{j}|^2 - \frac{|\mathbf{j}\mathbf{R}|^2}{R^2}\right)\frac{1}{R^2} \ , \tag{39}$$

and we can check the energy conservation $\partial W/\partial t + div\mathbf{S} = 0$.

It is worth computing the electromagnetic stress

$$\sigma_{ij} = \frac{1}{4\pi}\left[E_i E_j + H_i H_j - \frac{1}{2}\delta_{ij}(E^2 + H^2)\right] \ . \tag{40}$$

We take $\mathbf{j} = j\mathbf{e}_z$ and compute terms like $E_i(\omega)E_j(-\omega)$. Up to a common factor $-v^2\lambda^2 |j|^2 / 4\pi c^2$, we get

$$\begin{gathered} \sigma_{xx} = \frac{x^2(x^2+y^2)}{R^6} \ , \ \sigma_{yy} = \frac{y^2(x^2+y^2)}{R^6} \ , \ \sigma_{zz} = \frac{z^2(x^2+y^2)}{R^6} \ , \\ \sigma_{xy} = \frac{xy(x^2+y^2)}{R^6} \ , \ \sigma_{yz} = \frac{yz(x^2+y^2)}{R^6} \ , \ \sigma_{zx} = \frac{xz(x^2+y^2)}{R^6} \ . \end{gathered} \tag{41}$$

It is easy to check that the stress force $g_i = \partial\sigma_{ij}/\partial x_j$ is vanishing, as expected for average quantities, and the electromagnetic momentum $\mathbf{G} = \mathbf{S}/c^2$ in the equation of motion $\mathbf{g} = \partial\mathbf{G}/\partial t$ is a constant. The stress force acts only for time-dependent terms (containing factors like $e^{2i\omega t - 2i\lambda R}$), is oscillating in time and space, and produce a corresponding temporal change in the electromagnetic momentum (Poynting vector). The ponderomotive force in dielectrics, connected with the electromagnetic stress tensor, arises form the changes in the dielectric structure.

Near-field approximation. The electromagnetic potentials of the generalized dipole approximation, as given by equations (32) and (34), can also be used for moderate distances, with higher-order terms in the R'-expansion, if needed.

If $\lambda R \ll 1$ (the quasi-static approximation) the dipole fields are given by

$$\mathbf{E} = -\frac{iv}{\omega}\left[\mathbf{j} - 3\frac{(\mathbf{j}\mathbf{R})\mathbf{R}}{R^2}\right]\frac{1}{R^3} \ , \ \mathbf{H} = \frac{v}{c}\frac{\mathbf{j}\times\mathbf{R}}{R^3} \ , \tag{42}$$

where we recognize the dipole electric field and the magnetic field (the Biot-Savart law).

Finally, we comment upon the fields inside a polarizable body. From equations (4) it is obvious that the main contributions to the integrals of the electromagnetic potentials come from the region $\mathbf{R}' \simeq \mathbf{R}$. We get easily

$$\mathbf{A} \simeq \frac{2\pi a^2}{c}\mathbf{j} \ , \ \Phi = 0 \ , \tag{43}$$

where $\mathbf{j}$ is an average current, a is a cutoff length and we assumed $\lambda a \ll 1$. For a small body, a can be taken as the linear size of the body. The electric field (very small) is given by $\mathbf{E} = (2\pi i\lambda a^2/c)\mathbf{j}$ and the magnetic field is vanishing. Comparing with equations (42) we can see that the fields exhibit jumps at the surface of the body. In the equation of motion $(\omega^2 - \omega_c^2)u = -(q/m)E$ for the displacement u of the mobile charges with mass m, making use of $j = -i\omega nqu$, we get $(\omega^2 - \omega_c^2 + a^2\omega_p^2\omega^2/2c^2)u = 0$, where ω_c is a characteristic frequency and $\omega_p = \sqrt{4\pi nq^2/m}$

is the plasma frequency; hence, we can see a renormalization of the characteristic frequency $\omega = \omega_c/\sqrt{1+\omega_p^2/2\omega_0^2}$, where $\omega_0 = c/a$. This polarization resonance is active when the body is subject to an external excitation, and it can be seen as an iridiscence, as it is known for small bodies.