Journal of Theoretical Physics

Founded and Edited by M. Apostol **205** (2011)

ISSN 1453-4428

Scattering of longitudinal waves (sound) by small inhomogeneities in a fluid

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest Mg-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Abstract

The classical theory of scattering of longitudinal waves (sound) by small inhomogeneities (scatterers) in an ideal fluid is generalized to a distribution of scatterers and such as to include the effect of the inhomogeneities on the elastic properties of the fluid. The results are obtained by a new method of solving the wave equation with spatial restrictions (caused by the presence of the scatterers), which can also be applied to other types of inhomogeneities (like surface roughness, for instance). A coherent forward scattering is identified for a uniform distribution of scatterers (practically equivalent with a mean-field approach), which is due to the fact that our approach does not include multiple scattering. The reflected wave is obtained for a half-space of uniformly distributed carriers, as well as the field diffracted by a perfect lattice of scatterers.

Key words: scattering of sound; inhomogeneities; ideal fluid; reflected and diffracted longitudinal waves

PACS: 43.20.Fn; 43.20.El; 43.20.Bi; 47.35.Rs; 02.30.Jr

Highlights: A new model of inhomogeneities, including their effect on the elastic properties of the elastic medium. Longitudinal waves reflected from a half-space of uniformly distributed scatterers. Longitudinal waves (sound) diffracted by a perfect lattice of scatterers.

1 Introduction

The scattering of longitudinal waves (sound) by small inhomogeneities (scatterers) in an ideal fluid is a well-known subject (see, for instance, Ref. [1]). We derive here these classical results by a new method, which allows a generalization. The generalization consists in including the effect the inhomogeneities may have upon the elastic properties of the fluid localized on them (parameter η in this paper) and to get the scattered field arising from any distribution of scattering centers. The method can be applied also to other types of scatterers (like, for instance, a surface roughness). There is a great deal of interest today in scattering of sound, especially in random media (by using Foldy's theory and its recent developments),[2]-[4] and, in general, in complex media, where serious mathematical difficulties are encountered.[5] The main difficulty resides in formulating a convenient model of inhomogeneities, such as to allow for mathematically operational approaches. Though it does not include the multiple scattering, the model put forward here leads to definite results, such as the field reflected by a half-space of uniformly distributed scatterers, or the field diffracted by a perfect lattice of scatterers.

2 Background

We consider a homogeneous, isotropic, ideal fluid of infinite extension. A small displacement field $\mathbf{u}(\mathbf{r},\, t)$, where $\mathbf{r}$ denotes the position and t denotes the time, gives rise to a density imbalance $\delta n = -ndiv\mathbf{u}$ in the fluid density n, a local change of volume $\delta V = V div\mathbf{u}$ and a local change of pressure δp, depending on the equation of state of the fluid; for an adiabatic change, $\delta p = (\partial p/\partial n)_S \delta n = -n(\partial p/\partial n)_S div\mathbf{u}$, where S denotes the entropy. As it is well known,[1] such a fluid supports longitudinal waves (sound), described by the equation of motion

$$\frac{1}{c^2}\ddot{\mathbf{u}} - grad \cdot div\mathbf{u} = 0 \ , \tag{1}$$

where c is the sound velocity. Indeed, by taking the div in equation (1), we get the wave equation for free waves propagating with velocity c. The displacement field is subjected to the condition $curl\mathbf{u} = 0$. Therefore, it is convenient to introduce the potential function $\Phi = div\mathbf{u}$ (proportional to the pressure) and write equation (1) as

$$\frac{1}{c^2}\ddot{\Phi} - \Delta\Phi = 0 \ . \tag{2}$$

The sound propagation in fluids is also described by means of another potential function Ψ, defined by $\delta p = -\rho\partial\Psi/\partial t$ and $\mathbf{v} = \dot{\mathbf{u}} = grad\Psi$, where ρ is the (mass) density and $\mathbf{v}$ is the fluid velocity.[1] Then, Euler's equation $\rho\partial\mathbf{v}/\partial t + grad\delta p = 0$ (for small velocities $\mathbf{v}$) is satisfied identically, and the continuity equation $\partial\delta\rho/\partial t + \rho div\mathbf{v} = 0$ becomes the wave equation $\partial^2\Psi/\partial t^2 - c^2\Delta\Psi = 0$, through $\delta p = (\partial p/\partial\rho)_S\delta\rho$, with the sound velocity given by $c^2 = (\partial p/\partial\rho)_S$. The connection between the two potential function Ψ and Φ is given by

$$\delta p = -\rho\partial\Psi/\partial t = (\partial p/\partial\rho)_S\delta\rho = -\rho(\partial p/\partial\rho)_S div\mathbf{u} = -\rho c^2\Phi \ , \tag{3}$$

or

$$\frac{\partial\Psi}{\partial t} = c^2\Phi \ ; \tag{4}$$

for a monochromatic wave $\Psi = (ic^2/\omega)\Phi$. According to equation (1), the energy density (per unit mass) carried on by the longitudinal waves in a fluid is given by

$$e = \frac{1}{2}\dot{\mathbf{u}}^2 + \frac{1}{2}c^2\Phi^2 = \frac{1}{2}\frac{c^4}{\omega^2}(grad\Phi)^2 + \frac{1}{2}c^2\Phi^2 \ , \tag{5}$$

where equation (4) is used for a monochromatic wave. For a plane wave, equation (5) gives $e = c^2\Phi^2$.

3 Small inhomogenities

We assume a small inhomogeneity (foreign body, impurity) in an ideal fluid, placed at a fixed position $\mathbf{r}_i$, of a mean radius h_i (a scatterer). For h_i much smaller than the relevant wavelengths of the disturbances propagating in the fluid we can write the potential function Φ as

$$\Phi(\mathbf{r},\, t) = \varphi(\mathbf{r},\, t)\theta(|\mathbf{r}-\mathbf{r}_i| - h_i) \simeq \varphi(\mathbf{r},\, t)\theta(|\mathbf{r}-\mathbf{r}_i|) - h_i\varphi(\mathbf{r},\, t)\delta(|\mathbf{r}-\mathbf{r}_i|) \ , \tag{6}$$

where $\theta(x) = 1$ for $x > 0$, $\theta(x) = 0$ for $x < 0$ is the step function and δ is the Dirac function, or

$$\Phi = \varphi + \delta\Phi \ , \ \delta\Phi = -h_i\varphi(\mathbf{r}_i,\, t)\delta(|\mathbf{r}-\mathbf{r}_i|) \ . \tag{7}$$

The potential Φ satisfies the free wave equation (with specific boundary conditions at the surface of the inhomogeneity). According to our decomposition given by equation (6) we can see that φ satisfies the free wave equation in the whole space, while $\delta\Phi$ generates a source-term (a force), localized on the inhomogenity, which may give scattered waves. We introduce the potential Φ_1 for describing these scattered waves. It should obey the wave equation

$$\frac{1}{c^2}\ddot{\Phi}_1 - \Delta\Phi_1 = f \ , \tag{8}$$

where the force f is given by

$$f = \frac{1}{c^2}\delta\ddot{\Phi} - \Delta\delta\Phi \ . \tag{9}$$

Equation (8) is merely a re-writing of the wave equation for $\delta\Phi_0$. The force f is the difference between the inertial force $\delta\ddot{\Phi}/c^2$ and the elastic force $\Delta\delta\Phi$; it represents the distinct way the inhomogeneity responds to (follows) the wave motion in comparison with the fluid bulk. For waves localized on the inhomogeneity, equation (8) has the solution $\Phi_1 = \delta\Phi_0$. Another solutions are given by the waves scattered in the fluid by the inhomogeneity, *i.e.* waves generated in equation (8) by the source term f (a particular solution of equation (8)). We generalize this model of inhomogeneity by introducing a different "sound" velocity $\overline{c}$ in equation (9). The force is then written as

$$f = \frac{1}{\overline{c}^2}\delta\ddot{\Phi} - \Delta\delta\Phi \ . \tag{10}$$

Such a generalization amounts to assuming that the elastic properties of the fluid localized on the inhomogeneity are different than the elastic properties of the fluid bulk. For instance, the spatial variations of the scatterer shape may affect the elastic properties of the fluid in its neighbourhood. It is convenient to introduce the parameter $\eta = 1 - c^2/\overline{c}^2$ for describing such an "inhomogeneous" scatterer. A homogeneous scatterer (*i.e.*, the absence of the scatterer) would correspond to $\eta = 0$. A perfectly rigid scatterer would have $\overline{c} \rightarrow \infty$ and $\eta \rightarrow 1$.

Obviously, according to equations (6) and (7), the scheme of calculation put forwad here is a perturbation-theoretical scheme, with the mean radius h_i as the perturbation parameter. In view of the small magnitude of the mean radius h_i, we limit ourselves here to the first order of the perturbation theory.

We consider an incident plane wave $\varphi = \varphi_0 e^{-i\omega t + i\mathbf{kr}}$, where $\omega = ck$. Then, the source-term becomes

$$\delta\Phi = -h_i\varphi_0\delta(|\mathbf{r}-\mathbf{r}_i|)e^{-i\omega t + i\mathbf{kr}_i} \tag{11}$$

and the force given by equation (10) reads

$$f = h_i\varphi_0\left[\frac{\omega^2}{\overline{c}^2}\delta(|\mathbf{r}-\mathbf{r}_i|) - \Delta\delta(|\mathbf{r}-\mathbf{r}_i|)\right]e^{-i\omega t + i\mathbf{kr}_i} \ , \tag{12}$$

or

$$f = -\eta\frac{h_i\varphi_0\omega^2}{c^2}\delta(|\mathbf{r}-\mathbf{r}_i|)e^{-i\omega t + i\mathbf{kr}_i} + h_i\varphi_0\left[\frac{\omega^2}{c^2}\delta(|\mathbf{r}-\mathbf{r}_i|) - \Delta\delta(|\mathbf{r}-\mathbf{r}_i|)\right]e^{-i\omega t + i\mathbf{kr}_i} \ . \tag{13}$$

As it is well known, the solution of equation (8) is given by

$$\Phi_1(\mathbf{r}, t) = \frac{1}{4\pi}\int d\mathbf{r}'\frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}f(\mathbf{r}', t) \ , \tag{14}$$

with f given by equation (13). The second term in the *rhs* of equation (13) can be integrated by parts in equation (14), and we get the laplacean applied to the Green function (spherical wave) of the Helmholtz equation. This way, we get the localized waves

$$\Phi_{1l} = -h_i\varphi_0 \int d\mathbf{r}'\delta(\mathbf{r}-\mathbf{r}')\delta(|\mathbf{r}'-\mathbf{r}_i|) = -h_i\varphi_0\delta(|\mathbf{r}-\mathbf{r}_i|) \ , \tag{15}$$

which are precisely the localized waves $\Phi_{1l} = \delta\Phi$ given by equation (11), as expected (we leave aside the exponential factor $e^{-i\omega t+i\mathbf{k}\mathbf{r}_i}$). The first term in the *rhs* of equation (13) gives the scattered waves

$$\Phi_{1s} = -\eta\frac{h_i\varphi_0\omega^2}{4\pi c^2}e^{-i\omega t+i\mathbf{k}\mathbf{r}_i}\int d\mathbf{r}'\frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\delta(|\mathbf{r}'-\mathbf{r}_i|) \ . \tag{16}$$

We assume that the δ-function in equation (16) extends over the small distance h_i, *i.e.* $\delta(|\mathbf{r}-\mathbf{r}_i|) \simeq 1/h_i$ for $|\mathbf{r}-\mathbf{r}_i| < h_i$. Then, the integral in equation (16) is evaluated easily. We get

$$\Phi_{1s} \simeq -\eta\frac{v_i\varphi_0\omega^2}{4\pi c^2}e^{-i\omega t+i\mathbf{k}\mathbf{r}_i}\frac{e^{ik|\mathbf{r}-\mathbf{r}_i|}}{|\mathbf{r}-\mathbf{r}_i|} \ , \tag{17}$$

where v_i is the (mean) volume of the scatterer.

We can see that the validity of the perturbation-theoretical scheme requires $h_i \ll \lambda$, where $\lambda = c/\omega$ is the wavelength of the incident wave. According to equation (17), the scatterer generates spherical waves, the dfferential cross-section being given by

$$d\sigma = \eta^2\frac{v_i^2\omega^4}{(4\pi)^2c^4}d\Omega \ , \tag{18}$$

where Ω denotes the solid angle. As it is well known, it is proportional to the square volume of the scatterer and the fourth power of the frequency. The energy flux per unit mass) $\Gamma = cer^2d\Omega$ (with the origin of the reference frame at $\mathbf{r}_i$) is given by

$$\Gamma = \eta^2\frac{v_i^2\varphi_0^2\omega^4}{(4\pi)^2c}d\Omega \ , \tag{19}$$

which is to be compared with the energy flux $c^3\varphi_0^2$ per unit cross-sectional area in the incident wave (cross-section). The scattered field given by equation (17) does not exhibit directional effects, because the fluid velocity $\mathbf{v} = (ic^2/\omega)grad\delta\Phi$ in the source-term given by equation (11) is isotropic. We can say that the scattered field given by equation (17) arises from a "monopole" scatterer.

There is another solution of the free waves equation in the presence of a small inhomogenity placed at $\mathbf{r}_i$: it is given by $\Phi' \sim \delta(\mathbf{r}-\mathbf{r}_i)$. Indeed, it satisfies trivially the free waves equation for any $\mathbf{r} \neq \mathbf{r}_i$. This potential function should carry in front of the δ-function a factor proportional to the volume v_i. Since $2\pi h_i^2\delta(\mathbf{r}) = \delta(r)$, it is easy to see that this factor is $3v_i/2$. Under the action of an incident wave Φ_0 this solution changes by an amount which can be derived from $\partial\delta\Phi'\partial t = \mathbf{v}grad\Phi' = (3/2)v_i\mathbf{v}grad\delta(\mathbf{r}-\mathbf{r}_i)$, where $\mathbf{v} = grad\Psi_0$ is the velocity of the fluid particles. For a monochromatic wave, making use of $\Psi_0 = (ic^2/\omega)\Phi_0$, we get $\delta\Phi' = (3/2)v_i(c^2/\omega^2)grad\Phi_0 grad\delta(\mathbf{r}-\mathbf{r}_i)$, or, for a plane wave,

$$\delta\Phi' = \frac{3iv_ic^2\varphi_0}{2\omega^2}e^{-i\omega t+i\mathbf{k}\mathbf{r}_i}\mathbf{k}grad\delta(\mathbf{r}-\mathbf{r}_i) = \frac{3iv_i\varphi_0}{2k^2}e^{-i\omega t+i\mathbf{k}\mathbf{r}_i}\mathbf{k}grad\delta(\mathbf{r}-\mathbf{r}_i) \tag{20}$$

(the multiplication should be done for complex conjugate quantities). This change in the potential function gives rise to a force, similar with the force given above by equation (12). Introduced in equation (14), it generates a localized wave equal to $\delta\Phi'$, as expected, and a scattered wave given by

$$\Phi'_{1s} = -i\eta\frac{3v_i\varphi_0\omega^2}{8\pi c^2}e^{-i\omega t+i\mathbf{k}\mathbf{r}_i}\frac{\mathbf{k}}{k^2}\int d\mathbf{r}'\frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}grad\delta(|\mathbf{r}'-\mathbf{r}_i|) \ , \tag{21}$$

or

$$\Phi'_{1s} \simeq \eta\frac{3v_i\varphi_0\omega}{8\pi c}e^{-i\omega t+i\mathbf{k}\mathbf{r}_i}\frac{\mathbf{k}(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|}\frac{e^{ik|\mathbf{r}-\mathbf{r}_i|}}{|\mathbf{r}-\mathbf{r}_i|} \tag{22}$$

(leading approximation). We can see that these scattered waves exhibit directional effects, as arising from a "dipole" scatterer.

The total scattered field is obtained by adding equations (17) and (22). We get

$$\Phi_s = -\eta\frac{v_i\varphi_0 k}{4\pi}(k-\frac{3}{2}\mathbf{k}\mathbf{n}_i)e^{-i\omega t+i\mathbf{k}\mathbf{r}_i}\frac{e^{ik|\mathbf{r}-\mathbf{r}_i|}}{|\mathbf{r}-\mathbf{r}_i|} \ , \tag{23}$$

where $\mathbf{n}_i = (\mathbf{r}-\mathbf{r}_i)/|\mathbf{r}-\mathbf{r}_i|$ is the unit vector from the scatterer to the observation point. The cross section is given by

$$d\sigma = \eta^2\frac{v_i^2\omega^4}{(4\pi)^2c^4}(1-\frac{3}{2}\cos\theta_i)^2d\Omega \ , \tag{24}$$

where θ_i is the angle between the direction of propagation of the incident wave and the direction of observation from the scatterer. For a perfectly rigid scatterer we may take $\overline{c}\rightarrow\infty$ and $\eta\rightarrow 1$. It is worth noting that there is a scattering angle given by $\cos\theta = 2/3$ where the scattered field is vanishing. This is a well-known, classical result for the scattering of sound.[1]

4 Discussion and concluding remarks

For a distribution of inhomogeneities equation (23) gives the scattered field

$$\Phi_s = -\eta\frac{\varphi_0 k}{4\pi}e^{-i\omega t}\sum_i v_i(k-\frac{3}{2}\mathbf{k}\mathbf{n}_i)e^{i\mathbf{k}\mathbf{r}_i}\frac{e^{ik|\mathbf{r}-\mathbf{r}_i|}}{|\mathbf{r}-\mathbf{r}_i|} \ . \tag{25}$$

Let us assume a uniform distribution of identical scatterers ($v_i = v$), with a density σ, and take the origin as the observation point ($\mathbf{r} = 0$). The summation in equation (25) becomes an integral,

$$\Phi_s = -\eta\sigma v\frac{\varphi_0 k^2}{4\pi}e^{-i\omega t}\int d\mathbf{r}(1+\frac{3}{2}\cos\theta)e^{ikr\cos\theta}\frac{e^{ikr}}{r} \ , \tag{26}$$

where θ is the angle between the propagation vector $\mathbf{k}$ and the position $\mathbf{r}_i$ of the inhomogeneity. It is easy to see that the integral in equation (26) can be put in the form

$$\int d\mathbf{r}(1+\frac{3}{2}\cos\theta)e^{ikr\cos\theta}\frac{e^{ikr}}{r} = -\frac{\pi}{k^2}\int_{-1}^{1}du\frac{2+3u}{(1+u)^2} \ , \tag{27}$$

where $u = \cos\theta$. We can see that this integral has a singularity for $\theta = \pi$, arising from the backward scatterers (forward scattering). Indeed, we can see easily that for $\theta_i = \pi$ in equation (25) (a line of scatterers), we get a logarithmic singularity. This is an example of coherent forward scattering, corresponding to a vanishing phase $\mathbf{k}\mathbf{r}_i + kr_i = 0$ in equation (25), an expected result for a uniform distribution of scatterers without multiple scattering, which is equivalent with a mean-field approach for a uniform medium. It is to be compared with the scattering by one scatterer (placed at $\mathbf{r}_i = 0$), where the maximum of the scattered field lies in the backward direction.

This singularity arises from the fact that our approach does not include multiple scattering (for instance, forward and backward scattering).

Equation (25) gives reflected waves. Indeed, let us assume that we have a uniform distribution of identical scatterers in a half-space defined by $z > d$. The scattered field given by equation (25) can be written as

$$\Phi_s = -\eta v \frac{\varphi_0 k^2}{4\pi} e^{-i\omega t + i\mathbf{kr}} I \ , \tag{28}$$

where

$$I = -\frac{1}{2}\sum_i \frac{1}{r_i} e^{ikr_i(1+\cos\theta_i)} - \frac{3i}{2}\frac{\partial}{\partial k}\sum_i \frac{1}{r_i^2} e^{ikr_i(1+\cos\theta_i)} \ . \tag{29}$$

The summation in equation (29) is peformed oevr the half-space. It is convenient to introduce $\mathbf{k} = (\mathbf{k}_\perp, \kappa)$, where $\mathbf{k}_\perp$ is the wavevector parallel to the surface of the half-space and κ is the component of the wavevector perpendicular to this surface. It is also convenient to use cylindrical coordinates $\mathbf{r}_i = (\mathbf{r}_{i\perp}, z_i)$. The calculations are straightforward; they imply the known integral[7]

$$\int_{|z|} dr J_0(k_\perp \sqrt{r^2 - z^2}) e^{ikr} = \frac{i}{\kappa} e^{i\kappa|z|} \ . \tag{30}$$

We get the leading contribution to the scattered field

$$\Phi_s \simeq \eta\sigma v \frac{\varphi_0 k^2}{4\kappa^2} e^{-i\omega t + i\mathbf{k}_\perp \mathbf{r}_\perp} e^{-i\kappa z} \ , \tag{31}$$

which is the reflected field. The reflection coefficient (the ratio of the scattered amplitude to the amplitude of the incident wave) is $R = \eta\sigma v k^2/4\kappa^2$.

It is worth discussing a laticial distribution of identical scatterers. The force which generates the scattered field in this case contains a factor which has the latice periodicity. For instance, this force in equation (13) can be written as

$$-\eta \frac{h\varphi_0 \omega^2}{c^2} e^{i\mathbf{kr}} \sum_i \delta(|\mathbf{r} - \mathbf{r}_i|) e^{-i\mathbf{k}(\mathbf{r} - \mathbf{r}_i)} \tag{32}$$

(where the factor $e^{-i\omega t}$ is left aside). We can see that the summation over i is a periodic function of $\mathbf{r}$ with the lattice periodicity. Therefore, it can be expanded in a Fourier series involving only the reciprocal vectors $\mathbf{g}$ of the lattice. The scattered field given by equation (8) can also be expanded in a Fourier series of wavevectors $\mathbf{k} + \mathbf{g}$. We get the final result for the scattered field at large distances

$$\Phi_s = \eta\sigma v \varphi_0 \frac{e^{ikr}}{8\pi r} \sum_{\mathbf{g}} (k^2 + 3\mathbf{kg}) \int d\mathbf{r}' e^{i(\mathbf{k} - \mathbf{k}' + \mathbf{g})\mathbf{r}'} \ , \tag{33}$$

where $\mathbf{k}' = k\mathbf{r}/r$ is the wavevector of the scattered wave and integration is performed over the scatterers sample. We can see that the wave exhibits diffraction spots, provided the well-known Laue-Bragg diffraction condition $\mathbf{k} - \mathbf{k}' + \mathbf{g} = 0$ ($g^2 + 2\mathbf{kg} = 0$) is satisfied. The cross-section for a diffraction spot is given by $d\sigma = (\pi/8)(\eta\sigma v)^2 V (k^2 + 3\mathbf{kg})^2 d\Omega$, where V is the volume of the sample (compare with equation (24)). It can also be written as $d\sigma = (\pi/8)(\eta\sigma v k^2)^2 V (2 - 3\cos\theta)^2 d\Omega$, where θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$.

Finally, we may say that a new model of small inhomogeneities (scatterers) in an ideal fluid has been introduced here, which allows for including the effect the inhomogeneities may have on the elastic properties of the fluid (parameter η). The classical results for one scatterer have been re-derived by a new method of solving the wave equation and the wave reflected by a half-space of uniformly distributed scatterers, as well as the wave diffracted by a perfect lattice of scatterers have been derived. The model can be extended to other types of inhomogeneities, like, for instance, a rough surface, and may also be useful in the complex problem of multiple scattering.

Acknowledgments. The author is indebted to the members of the Laboratory of Theoretical Physics at Magurele-Bucharest for many useful discussions and a throughout checking of this work.

References

[1] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics*, vol. 6, *Fluid Mechanics*, Elsevier, Oxford, 2002.

[2] L. L. Foldy, "The multiple scattering of waves. I. General theory of isotropic scattering by randomly distributed scatterers", Phys. Rev. **67** 107-119 (1945).

[3] M. C. W. van Rossum and T. M. Nieuwenhuizen, "Multiple scattering of classical waves: microscopy, mesoscopy and diffusion", Revs. Mod. Phys. **71** 313-371 (1999)).

[4] W. E. Ostashev, "Sound propagation and scattering in media with random inhomogeneities of sound speed, density and medium velocity", Waves in Random Media, **4** 403-428 (1994)).

[5] R. H. Hackman, "Acoustic scattering from elastic solids", Physical Acoustics **22** 1-194 (1993).

[6] See, for instance, M. Ochmann, "Full-field equations for acoustic radiation and scattering", J. Ac. Soc. Am. **105** 2574-2584 (1999).

[7] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, eds. A. Jeffrey and D. Zwillinger, 6th edition, Academic Press, San Diego, 2000, pp. 714-715, 6.677.1,2.