Journal of Theoretical Physics

Founded and Edited by M. Apostol **206** (2011)

ISSN 1453-4428

On the molecular forces acting between macroscopic bodies

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Abstract

The eigenfrequencies are identified for two electromagnetically-coupled semi-infinite solids with plane-parallel surfaces (two half-spaces) separated by a third, polarizable body. The corresponding van der Waals-London and Casimir forces are calculated from the zero-point energy (vacuum fluctuations) of the normal modes. It is shown how the results can be extended to bodies of any shape, in particular to a sphere interacting with a half-space. The calculations are performed by using the well-known Lorentz-Drude (plasma) model of (non-magnetic) polarizable matter. The polarization degrees of freedom are explicitly introduced. It is shown that the polarization dynamical variables for the two bodies are coupled through the electromagnetic field, very similar with two infnite sets of coupled harmonic oscillators.

PACS: 03.50.De; 41.20.-q; 41.20.Jb; 52.25.Mq; 42.50.Wk

Keywords: *molecular forces; van der Waals-London force; Casimir force; matter polarization; vacuum fluctuations*

1 Introduction

As it is well known, the molecular forces acting between atoms (molecules), known as van der Waals-London and Casimir forces, have been derived originally by quantum-mechanical calculations in the non-retarded (small distance)[1]-[3] and, respectively, retarded (large distance) regime[4] (see also Refs. [5, 6]). The force acting in the retarded regime between an atom and a semi-infinite conductor (half-space) has also been derived by quantum-mechanical calculations,[4] while the retarded force acting between two conducting half-spaces (Casimir force) has been originally derived by advancing arguments related to the zero-point energy (vacuum fluctuations) of the electromagnetic field with suitable boundary conditions at the surfaces of the two half-spaces.[7] On the other hand, it was realized that these molecular forces are related to the internal electrical polarization of matter. The macroscopic bodies bring their own characteristics with respect to the electrical polarization (like plasmons, polaritons, surface effects, etc), in comparison with individual quantum particles.[8]-[12]

Molecular forces acting between macroscopic bodies, either conductors or dielectrics, have been derived by the theory of the quantum-statistical electromagnetic fluctuations,[13]-[15] as well as within the framework of the source theory.[16, 17] Both theories consider, on one hand, the polarization as an external source, and estimate the response of the electromagnetic field to this source, and, on the other hand, include polarization (via the dielectric function) in the electromagnetic field, viewing the latter as a dynamical variable (coordinate). For this reason, there was never clearly grasped which are the normal modes which fluctuate and bring the zero-point energy in the molecular forces.

We describe here the polarization by a displacement field of the mobile charges in polarizable matter and solve the coupled equations of motion of this field, interacting via the electromagnetic field, for two semi-infinite solids with plane-parallel surfaces (two half-spaces) separated by a third, polarizable body. The calculations are done by using the well-known Lorentz-Drude (plasma) model of (non-magnetic) polarizable matter. We show that the polarizations of the two bodies interact with each other via their electromagnetic field, very much alike two infinite sets of coupled harmonic oscillators. The normal modes of the ensemble of the two bodies are identified and the eigenfrequencies are computed. The force is derived from the zero-point energy (vacuum fluctuations) of these normal modes. We compute the van der Waals-London and Casimir forces for two half-spaces, either conductors or dielectrics, separated, in general, by a third polarizable body. In view of the great deal of interest developed recently for the subject,[18]-[37] we show here how to compute such forces between bodies of any shape, and give the result for the force acting between a sphere and a half-space.

Some particular results concerning the derivation of the molecular forces along the lines described above have been previously published.[38, 39] The method used here has also been previously illustrated in Refs. [40, 41].

2 Matter polarization

We adopt a generic model of matter polarization consisting of identical mobile charges q, with mass m and density n, moving in a rigid, neutralizing background of volume V. A small displacement field $\mathbf{u}(\mathbf{R},t)$ in the position $\mathbf{R}$ of these charges gives, at the time t, a local density imbalance $\delta n = -n div\mathbf{u}$ and a polarization charge density $\rho = -nq div\mathbf{u}$. We can see that $\mathbf{P} = nq\mathbf{u}$ is the polarization. Therefore, the displacement field $\mathbf{u}(\mathbf{R},t)$ is a representation for the polarization field $\mathbf{P}(\mathbf{R},t)$. The displacement field obeys the Newton law of motion

$$m\ddot{\mathbf{u}} = q(\mathbf{E}+\mathbf{E}_0) - m\omega_c^2\mathbf{u} - m\gamma\dot{\mathbf{u}} \ , \tag{1}$$

where $\mathbf{E}$ is the polarization electric field generated by the polarization charges (and currents), ω_c is a characteristic frequency, γ is a (small) damping factor and $\mathbf{E}_0$ is an external electric field. This is the well-known Lorentz-Drude (plasma) model of polarizable matter,[42]-[44] which assumes a homogeneous, isotropic matter, without spatial dispersion, represented by a field of harmonic oscillators of frequency ω_c. Taking the temporal Fourier transform of equation (1), with $\mathbf{E}_t = \mathbf{E}+\mathbf{E}_0$ the total electric field, we get the electric susceptibility $\chi(\omega) = P/E_t$ and the dielectric function

$$\varepsilon(\omega) = 1 + 4\pi\chi(\omega) = \frac{\omega^2-\omega_c^2-\omega_p^2}{\omega^2-\omega_c^2+i\omega\gamma} = \frac{\omega^2-\omega_L^2}{\omega^2-\omega_T^2+i\omega\gamma} \ , \tag{2}$$

where $\omega_p = \sqrt{4\pi nq^2/m}$ is the plasma frequency. This is also well known as the Lydane-Sachs-Teller dielectric function,[45] with the longitudinal frequency $\omega_L = \sqrt{\omega_c^2+\omega_p^2}$ and the transverse frequency $\omega_T = \omega_c$. The latter can be taken as the main absorption frequency of the substance. The model can be generalized by including the spatial dispersion, several characteristic frequencies ω_c, or by adding an external magnetic field, etc. It is worth noting the absence of the magnetic part of the Lorentz force in equation (1), according to the non-relativistic motion of the slight displacement $\mathbf{u}$. It is easy to see that, apart from relativistic contributions, it would introduce non-linearities in equation (1), which are beyond our assumption of a small displacement $\mathbf{u}$. Using spatial Fourier transforms, this approximation can be formulated as $\mathbf{K}\mathbf{u}(\mathbf{K}) \ll 1$, where $\mathbf{K}$ is the wavevector.

In general, an additional displacement $\mathbf{u}_0$ can be introduced in such a model, originating in external causes, subjected to collisions and obeying a different, averaged equation of motion, $m\dot{\mathbf{u}}_0 = q\mathbf{E}_t\tau$, where τ is a relaxation time; as it is well known, it gives rise to a density of "conduction" current $\mathbf{j}_0 = nq\dot{\mathbf{u}}_0 = (nq^2\tau/m)\mathbf{E}_t$ and the conductivity $\sigma = nq^2\tau/m$. We can see that it implies $\omega_c = 0$ in equation (1), a condition which defines the conductors; for dielectrics, $\omega_c \neq 0$. We leave aside the conduction current $\mathbf{j}_0$. We also leave aside the small damping parameter γ in the equation of motion (1).

The displacement field $\mathbf{u}$ produces polarization charge and current densities given by

$$\rho = -div\mathbf{P} = -nqdiv\mathbf{u} \ , \ \mathbf{j} = \frac{\partial \mathbf{P}}{\partial t} = nq\dot{\mathbf{u}} \ , \tag{3}$$

which can be used to compute the electromagnetic potentials

$$\begin{gathered} \Phi(\mathbf{R}, t) = \int d\mathbf{R}' \frac{\rho(\mathbf{R}', t - |\mathbf{R} - \mathbf{R}'|/c)}{|\mathbf{R} - \mathbf{R}'|} \ , \\ \mathbf{A}(\mathbf{R}, t) = \frac{1}{c} \int d\mathbf{R}' \frac{\mathbf{j}(\mathbf{R}', t - |\mathbf{R} - \mathbf{R}'|/c)}{|\mathbf{R} - \mathbf{R}'|} \end{gathered} \tag{4}$$

(subjected to the Lorenz gauge $div\mathbf{A} + (1/c)\partial\Phi/\partial t = 0$). These potentials give rise to the electric field $\mathbf{E}$ in equation (1), whence we can get the displacement $\mathbf{u}$. This way, we can compute the electromagnetic fields of a polarizable body, subjected to the action of an external electromagnetic field.

3 Half-space

For a half-space extending over the region $z > d$ we take the polarization as

$$\mathbf{P} = nq(\mathbf{u}, \, u_z)\theta(z - d) \ , \tag{5}$$

where $\theta(z) = 0$ for $z < 0$ and $\theta(z) = 1$ for $z > 0$ is the step function. The polarization charge and current densities are given by

$$\begin{gathered} \rho = -nq(div\mathbf{u} + \frac{\partial u_z}{\partial z})\theta(z - d) - nqu_z(d)\delta(z - d) \ , \\ \mathbf{j} = nq(\dot{\mathbf{u}}, \, \dot{u}_z)\theta(z - d) \ . \end{gathered} \tag{6}$$

We use Fourier decompositions of the type

$$\mathbf{u}(\mathbf{r}, z; t) = \frac{1}{2\pi} \sum_{\mathbf{k}} \int d\omega \mathbf{u}(\mathbf{k}, z; \omega) e^{-i\omega t + i\mathbf{k}\mathbf{r}} \ , \tag{7}$$

where $\mathbf{R} = (\mathbf{r}, \, z)$, and may omit ocassionally the arguments $\mathbf{k}$, ω, writing simply $\mathbf{u}(z)$, or $\mathbf{u}$. The electromagnetic potentials given by equations (4) includes the "retarded" Coulomb potential $e^{i\frac{\omega}{c}|\mathbf{R} - \mathbf{R}'|} / \left| \mathbf{R} - \mathbf{R}' \right|$, for which we use the well-known decomposition[46]

$$\frac{e^{i\lambda|\mathbf{R} - \mathbf{R}'|}}{|\mathbf{R} - \mathbf{R}'|} = \frac{i}{2\pi} \int d\mathbf{k} \frac{1}{\kappa} e^{i\mathbf{k}(\mathbf{r} - \mathbf{r}')} e^{i\kappa|z - z'|} \ , \tag{8}$$

where $\lambda = \omega/c$ and $\kappa = \sqrt{\lambda^2 - k^2}$. It is more convenient to compute first the vector potential $\mathbf{A}$ and then derive the scalar potential Φ from the gauge equation $div\mathbf{A} - i\lambda\Phi = 0$. The calculations are straightforward and we get the Fourier tranforms of the potentials

$$\Phi(\mathbf{k}, z; \omega) == \tfrac{2\pi}{\kappa} \int_d^\infty dz' \mathbf{k}\mathbf{u} e^{i\kappa|z-z'|} - \tfrac{2\pi i}{\kappa} \tfrac{\partial}{\partial z} \int_d^\infty dz' u_z e^{i\kappa|z-z'|} \ ,$$
$$\mathbf{A}(\mathbf{k}, z; \omega) = \tfrac{2\pi\lambda}{\kappa} \int_d^\infty dz' (\mathbf{u}, \, u_z) e^{i\kappa|z-z'|} \tag{9}$$

(where we have left aside the factor nq; it is restored in the final formulae). In order to compute the electric field ($\mathbf{E} = i\lambda\mathbf{A} - grad\Phi$) it is convenient to refer the in-plane vectors (*i.e.*, vectors parallel with the surface of the half-space) to the vectors $\mathbf{k}$ and $\mathbf{k}_\perp = e_z \times \mathbf{k}$, where e_z is the unit vector along the z-direction; for instance, we write

$$\mathbf{u} = u_1 \frac{\mathbf{k}}{k} + u_2 \frac{\mathbf{k}_\perp}{k} \tag{10}$$

and a similar representation for the electric field parallel with the surface of the half-space. In performing the calculations, it is worth paying attention to the derivative of the modulus function, according to the equation

$$\frac{\partial^2}{\partial z^2} e^{i\kappa|z-z'|} = -\kappa^2 e^{i\kappa|z-z'|} + 2i\kappa\delta(z - z') \ . \tag{11}$$

We get the electric field

$$E_1 = 2\pi i\kappa \int_d^\infty dz' u_1 e^{i\kappa|z-z'|} - \tfrac{2\pi k}{\kappa} \tfrac{\partial}{\partial z} \int_d^\infty dz' u_z e^{i\kappa|z-z'|} \ ,$$
$$E_2 = \tfrac{2\pi i\lambda^2}{\kappa} \int_d^\infty dz' u_2 e^{i\kappa|z-z'|} \ , \tag{12}$$
$$E_z = -\tfrac{2\pi k}{\kappa} \tfrac{\partial}{\partial z} \int_d^\infty dz' u_1 e^{i\kappa|z-z'|} + \tfrac{2\pi i k^2}{\kappa} \int_d^\infty dz' u_z e^{i\kappa|z-z'|} - 4\pi u_z \theta(z - d) \ .$$

Making use of equations (12), we can check easily the equalities

$$ikE_1 + \frac{\partial E_z}{\partial z} = -4\pi \left(iku_1 + \frac{\partial u_z}{\partial z} \right) \theta(z - d) - 4\pi u_z(z = d)\delta(z - d) \ , \tag{13}$$

which is an expression of Gauss's law, and

$$k\frac{\partial E_1}{\partial z} + i\kappa^2 E_z = -4\pi i\lambda^2 u_z \theta(z - d) \ , \tag{14}$$

which reflects Faraday's and Maxwell-Ampere's equations. From equation (13), we can check the transversality condition $div\mathbf{E} = 0$ for the electric field outside the half-space ($z < d$).

We use now the equation of motion (1) (with $\gamma = 0$) for E_2 given by equation (12) and for the combinations $iku_1 + \partial u_z/\partial z$ and $k\partial u_1/\partial z + i\kappa^2 u_z$ in the region $z > d$. Taking into account that $div\mathbf{E}_0 = 0$ and $k\partial E_{01}/\partial z + i\kappa^2 E_{0z} = 0$ (for a plane wave) we get

$$iku_1 + \frac{\partial u_z}{\partial z} = 0 \ , \ k\frac{\partial u_1}{\partial z} + i\kappa'^2 u_z = 0 \ ,$$

or

$$\tfrac{\partial^2 \mathbf{u}}{\partial z^2} + \kappa'^2 \mathbf{u} = 0 \ , \tag{15}$$

where

$$\kappa'^2 = \kappa^2 - \frac{\lambda^2 \omega_p^2}{\omega^2 - \omega_c^2} \ . \tag{16}$$

The components $u_{1,2}$ of the displacement field are given by $u_{1,2} = A_{1,2}e^{i\kappa' z}$, where $A_{1,2}$ are constants, while $u_z = -(k/\kappa')A_1 e^{i\kappa' z}$ (we restrict ourselves to outgoing waves, $\kappa' > 0$). The total electric field inside the half-space is given by the equation of motion (1):

$$\mathbf{E}_t = -\frac{m}{q}(\omega^2 - \omega_c^2)\mathbf{u} \tag{17}$$

for $z > d$. We can see that the field propagates in the half-space with a modified wavevector κ', according to the Ewald-Oseen extinction theorem.[47] The modified wavevector κ' given by equation (16) can also be written as

$$\kappa'^2 = \varepsilon\frac{\omega^2}{c^2} - k^2 \ , \tag{18}$$

where ε is the dielectric function (as given by equation (2)). We can check the well-known polaritonic dispersion relation $\varepsilon\omega^2 = c^2K'^2$, where $\mathbf{K}' = (\mathbf{k}, \kappa')$ is the wavevector.

The amplitudes $A_{1,2}$ can be derived from the original equation (1) and the field equations (12) (for $z > d$). We get

$$\begin{aligned} &\tfrac{1}{2}A_1\omega_p^2\tfrac{\kappa\kappa'+k^2}{\kappa'(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{i\kappa z} = \tfrac{q}{m}E_{01} \ , \\ &\tfrac{1}{2}A_2\omega_p^2\tfrac{\lambda^2}{\kappa(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{i\kappa z} = \tfrac{q}{m}E_{02} \ . \end{aligned} \tag{19}$$

The (polarization) electric field, both inside and outside the half-space, can be computed from equations (12). We get

$$\begin{aligned} E_1 &= -4\pi nqA_1\tfrac{\omega^2-\omega_c^2}{\omega_p^2}e^{i\kappa' z} - 2\pi nqA_1\tfrac{\kappa\kappa'+k^2}{\kappa'(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{i\kappa z} \ , \ z > d \ , \\ E_2 &= -4\pi nqA_2\tfrac{\omega^2-\omega_c^2}{\omega_p^2}e^{i\kappa' z} - 2\pi nqA_2\tfrac{\lambda^2}{\kappa(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{i\kappa z} \ , \ z > d \ , \\ E_z &= 4\pi nqA_1\tfrac{k(\omega^2-\omega_c^2)}{\kappa'\omega_p^2}e^{i\kappa' z} + 2\pi nqA_1\tfrac{k(\kappa\kappa'+k^2)}{\kappa\kappa'(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{i\kappa z} \ , \ z > d \ . \end{aligned} \tag{20}$$

for $z > d$. It is worth noting that the polarization electric field, as given by equations (20), includes both the external field $\sim e^{i\kappa z}$ (with opposite sign) and the displacement field $\mathbf{u} \sim e^{i\kappa' z}$. This can be checked easily by using equations (19) and (20). The (polarization) electric field outside the half-space (in the region $z < d$) is given by

$$\begin{aligned} E_1 &= -2\pi nqA_1\tfrac{\kappa\kappa'-k^2}{\kappa'(\kappa'+\kappa)}e^{i(\kappa'+\kappa)d}e^{-i\kappa z} \ , \ z < d \ , \\ E_2 &= -2\pi nqA_2\tfrac{\lambda^2}{\kappa(\kappa'+\kappa)}e^{i(\kappa'+\kappa)d}e^{-i\kappa z} \ , \ z < d \end{aligned} \tag{21}$$

and $E_z = (k/\kappa)E_1$ for $z < d$. We can see that it is the field reflected by the half-space ($\kappa \rightarrow -\kappa$). Making use of equations (19) and (21) we get the total electric field $\mathbf{E}_t = \mathbf{E} + \mathbf{E}_0$ outside the half-space

$$\begin{aligned} E_{t1} &= -2\pi nqA_1\tfrac{\kappa\kappa'-k^2}{\kappa'(\kappa'+\kappa)}e^{i(\kappa'+\kappa)d}e^{-i\kappa z} + 2\pi nqA_1\tfrac{\kappa\kappa'+k^2}{\kappa'(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{i\kappa z} \ , \\ E_{t2} &= -2\pi nqA_2\tfrac{\lambda^2}{\kappa(\kappa'+\kappa)}e^{i(\kappa'+\kappa)d}e^{-i\kappa z} + 2\pi nqA_2\tfrac{\lambda^2}{\kappa(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{i\kappa z} \ , \\ E_{tz} &= -2\pi nqA_1\tfrac{k(\kappa\kappa'-k^2)}{\kappa\kappa'(\kappa'+\kappa)}e^{i(\kappa'+\kappa)d}e^{-i\kappa z} - 2\pi nqA_1\tfrac{k(\kappa\kappa'+k^2)}{\kappa\kappa'(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{i\kappa z} \end{aligned} \tag{22}$$

for $z < d$.

The magnetic field, given by $\mathbf{H} = curl\mathbf{A}$, can be obtained from equation (9) for the vector potential. It is given by

$$
\begin{array}{c}
H_1 = 4\pi nqA_2 \frac{\kappa'(\omega^2-\omega_c^2)}{\lambda\omega_p^2} e^{i\kappa' z} + 2\pi nqA_2 \frac{\lambda}{\kappa'-\kappa} e^{i(\kappa'-\kappa)d} e^{i\kappa z} \ , \ z > d \ , \\
H_2 = -4\pi nqA_1 \frac{\lambda(\omega^2-\omega_p^2-\omega_c^2)}{\kappa'\omega_p^2} e^{i\kappa' z} - 2\pi nqA_1 \frac{\lambda(\kappa\kappa'+k^2)}{\kappa\kappa'(\kappa'-\kappa)} e^{i(\kappa'-\kappa)d} e^{i\kappa z} \ , \ z > d \ , \\
H_z = -4\pi nqA_2 \frac{k(\omega^2-\omega_c^2)}{\lambda\omega_p^2} e^{i\kappa' z} - 2\pi nqA_2 \frac{\lambda k}{\kappa(\kappa'-\kappa)} e^{i(\kappa'-\kappa)d} e^{i\kappa z} \ , \ z > d
\end{array} \tag{23}
$$

for $z > d$ and

$$
\begin{array}{c}
H_1 = -2\pi nqA_2 \frac{\lambda}{\kappa'+\kappa} e^{i(\kappa'+\kappa)d} e^{-i\kappa z} \ , \ z < d \ , \\
H_2 = 2\pi nqA_1 \frac{\lambda(\kappa\kappa'-k^2)}{\kappa\kappa'(\kappa'+\kappa)} e^{i(\kappa'+\kappa)d} e^{-i\kappa z} \ , \ z < d \ , \\
H_z = -2\pi nqA_2 \frac{\lambda k}{\kappa(\kappa'+\kappa)} e^{i(\kappa'+\kappa)d} e^{-i\kappa z} \ , \ z < d
\end{array} \tag{24}
$$

for $z < d$. We can check the Maxwell equation $curl\mathbf{E} = i\lambda\mathbf{H}$. Making use of equations (19) and (23), from $curl\mathbf{E}_0 = i\lambda\mathbf{H}_0$ we get the total magnetic field $\mathbf{H}_t = \mathbf{H} + \mathbf{H}_0$ inside the half-space

$$
\begin{array}{c}
H_{t1} = 4\pi nqA_2 \frac{\kappa'(\omega^2-\omega_c^2)}{\lambda\omega_p^2} e^{i\kappa' z} \ , \ H_{t2} = -4\pi nqA_1 \frac{\lambda(\omega^2-\omega_p^2-\omega_c^2)}{\kappa'\omega_p^2} e^{i\kappa' z} \ , \\
H_{tz} = -4\pi nqA_2 \frac{k(\omega^2-\omega_c^2)}{\lambda\omega_p^2} e^{i\kappa' z}
\end{array} \tag{25}
$$

for $z > d$ and the total magnetic field outside the half-space

$$
\begin{array}{c}
H_{t1} = -2\pi nqA_2 \frac{\lambda}{\kappa'+\kappa} e^{i(\kappa'+\kappa)d} e^{-i\kappa z} - 2\pi nqA_2 \frac{\lambda}{\kappa'-\kappa} e^{i(\kappa'-\kappa)d} e^{i\kappa z} \ , \\
H_{t2} = 2\pi nqA_1 \frac{\lambda(\kappa\kappa'-k^2)}{\kappa\kappa'(\kappa'+\kappa)} e^{i(\kappa'+\kappa)d} e^{-i\kappa z} + 2\pi nqA_1 \frac{\lambda(\kappa\kappa'+k^2)}{\kappa\kappa'(\kappa'-\kappa)} e^{i(\kappa'-\kappa)d} e^{i\kappa z} \ , \\
H_{tz} = -2\pi nqA_2 \frac{\lambda k}{\kappa(\kappa'+\kappa)} e^{i(\kappa'+\kappa)d} e^{-i\kappa z} + 2\pi nqA_2 \frac{\lambda k}{\kappa(\kappa'-\kappa)} e^{i(\kappa'-\kappa)d} e^{i\kappa z}
\end{array} \tag{26}
$$

for $z < d$.

The amplitudes $A_{1,2}$ can be viewed either as being determined by the external field $\mathbf{E}_0$ (and $\mathbf{H}_0$) through equations (19), or as free parameters. In the latter case equations (19) are not valid anymore, but the (polarization) electric and magnetic fields given by equations (20), (21), (23) and (24) hold. We can check also that all the fields are continuous at the surface $z = d$, except for E_z and E_{tz}, which exhibit a discontinuity ($E_{tz}(z = d^-) = \varepsilon E_{tz}(z = d^+)$), as expected.

4 Energy and momentum conservation. Stress force

Making use of the equation of motion (1) and Maxwell equations

$$
curl\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \ , \ curl\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} nq\dot{\mathbf{u}} \ , \tag{27}
$$

we get easily the well-known law of energy conservation[48]

$$
\frac{\partial}{\partial t}(E^+_{em} + E_m) = -\int d\mathbf{f}\mathbf{S}^+ \ , \tag{28}
$$

where

$$E_{em}^{+}=\frac{1}{8\pi}\int d\mathbf{R}(|\mathbf{E}|^{2}+|\mathbf{H}|^{2}) \qquad (29)$$

is the energy of the electromagnetic field inside the half-space,

$$E_{m}=\frac{1}{2}nm\int d\mathbf{R}(|\dot{\mathbf{u}}|^{2}+\omega_{c}^{2}|\mathbf{u}|^{2}) \qquad (30)$$

is the matter energy and

$$\mathbf{S}^{+}=\frac{c}{8\pi}\mathbf{E}^{*}\times\mathbf{H}+c.c. \qquad (31)$$

is the Poynting vector. The integration in equations (29) and (30) is performed over the volume of the half-space and the integration in equation (28) is performed over the surface of the half-space (oriented to the outside), $d\mathbf{f}$ denoting the infinitesimal element of area. For instance, for an infinite half-space the surface $\mathbf{f}$ is given by $z=d$, oriented toward negative values of z (we assume that the field is vanishing at infinity). Similarly, the energy conservation reads

$$\frac{\partial}{\partial t}E_{em}^{-}=-\int d\mathbf{f}\mathbf{S}^{-} \qquad (32)$$

for the energy of the electromagnetic field outside the half-space. Since the in-plane components of the field are continuous at the surface $z=d$ (or any other surface placed at a finite distance) the two (oriented) Poynting vectors cancel each out. Their contribution is also vanishing for any other surface placed at infinity, so we get the conservation of the total energy

$$\frac{\partial}{\partial t}(E_{em}^{+}+E_{em}^{-}+E_{m})=0\ . \qquad (33)$$

It is also well known the law of momentum conservation[48]

$$f_{i}+\frac{\partial}{\partial t}G_{i}^{+}=\partial_{j}\sigma_{ij}^{+}\ ,\ \frac{\partial}{\partial t}G_{i}^{-}=\partial_{j}\sigma_{ij}^{-} \qquad (34)$$

where $\mathbf{f}$ is the Lorentz force acting upon charges and currents in matter, $\mathbf{G}=\mathbf{S}/c^{2}$ is the electromagnetic momentum and

$$\sigma_{ij}=\frac{1}{8\pi}\left[E_{i}^{*}E_{j}+H_{i}^{*}H_{j}-\frac{1}{2}\delta_{ij}(|\mathbf{E}|^{2}+|\mathbf{H}|^{2})\right]=c.c. \qquad (35)$$

is the stress tensor (whose components are labelled by i, j). It is easy to see that the total force is given by the surface integral $\int df_{j}\sigma_{ij}$, For surfaces placed at infinity this force is vanishing, but it gives a non-vanishing contribution on the surface $z=d$ (σ_{zz}), arising from the discontinity

$$\Delta(E_{z}^{2})=\left|E_{z}(z=d^{-})\right|^{2}-\left|E_{z}(z=d^{+})\right|^{2}\ ; \qquad (36)$$

for instance, taking the average values with respect to time by $(1/T)\int dt$, where $T\rightarrow\infty$, and making use of equations (17) and (22) for the Fourier transforms of the (total) field, this discontinuity is proportional to

$$\frac{k^{2}}{\kappa'^{2}}\frac{\varepsilon+1}{\varepsilon-1}\left|A_{1}\right|^{2}\ . \qquad (37)$$

A similar discontinuity occurs also in the absence of the external field. At first sight, it seems that the half-space would experience a stress force acting upon its surface on behalf of the electromagnetic field, even in the absence of an external electromagnetic field. Such a force (in the context of the electromagnetic fluctuations) is usually associated with the Casimir (and van der Waals-London) forces.[13]-[17] It is due to the infinite extension of the half-space along the z-direction. In fact, such bodies have a finite extension along the z-axis, so that there is another surface where the discontinuity given by equation (36) is cancelled out. In general, the stress forces are internal forces; they act on the surface of the bodies (or interfaces of inhomogeneous bodies), giving a surface tension.

5 Two half-spaces

For a half-space extending in the region $z < -d$ we can repeat the calculations done in the previous **Section**. The displacement field in this case is written as $(\mathbf{v},\, v_z)\theta(-z-d)$. It is easy to see that we can get the results for the half space extending in the region $z < -d$ from those pertaining to the half-space extending in the region $z > d$ by changing z into $-z$. For instance, the displacement field is given by $v_{1,2} = B_{1,2}e^{-i\kappa' z}$ and $v_z = (k/\kappa')B_1 e^{-i\kappa' z}$, where $B_{1,2}$ are constant amplitudes; the electric field is given by

$$\begin{aligned}
E_1 &= -4\pi nqB_1\tfrac{\omega^2-\omega_c^2}{\omega_p^2}e^{-i\kappa' z} - 2\pi nqB_1\tfrac{\kappa\kappa'+k^2}{\kappa'(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{-i\kappa z}\ ,\ z<-d\ ,\\
E_2 &= -4\pi nqB_2\tfrac{\omega^2-\omega_c^2}{\omega_p^2}e^{-i\kappa' z} - 2\pi nqB_2\tfrac{\lambda^2}{\kappa(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{-i\kappa z}\ ,\ z<-d\ ,\\
E_z &= -4\pi nqB_1\tfrac{k(\omega^2-\omega_c^2)}{\kappa'\omega_p^2}e^{-i\kappa' z} - 2\pi nqB_1\tfrac{k(\kappa\kappa'+k^2)}{\kappa\kappa'(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{-i\kappa z}\ ,\ z<-d
\end{aligned} \tag{38}$$

for $z < -d$ and

$$\begin{aligned}
E_1 &= -2\pi nqB_1\tfrac{\kappa\kappa'-k^2}{\kappa'(\kappa'+\kappa)}e^{i(\kappa'+\kappa)d}e^{i\kappa z}\ ,\ z>-d\ ,\\
E_2 &= -2\pi nqB_2\tfrac{\lambda^2}{\kappa(\kappa'+\kappa)}e^{i(\kappa'+\kappa)d}e^{i\kappa z}\ ,\ z>-d
\end{aligned} \tag{39}$$

and $E_z = -(k/\kappa)E_1$ for $z > -d$; and the amplitudes $B_{1,2}$ are given by

$$\begin{aligned}
\tfrac{1}{2}B_1\omega_p^2\tfrac{\kappa\kappa'+k^2}{\kappa'(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{-i\kappa z} &= \tfrac{q}{m}E_{01}\ ,\\
\tfrac{1}{2}B_2\omega_p^2\tfrac{\lambda^2}{\kappa(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{-i\kappa z} &= \tfrac{q}{m}E_{02}\ .
\end{aligned} \tag{40}$$

We consider now two half-spaces, one, denoted by 1, extending in the region $z > d/2$, another, denoted by 2, occupying the region $z < -d/2$. The field pertaining to these half-spaces is given here and in the previous **Section**, with d replaced by $d/2$. We focus on the amplitudes equations (19) and (40). The external field for the half-space 2 (equations (40)) is the field given by equations (21), produced by half-space 1 in the region $z < d/2$; similarly, the external field for the half-space 1 (equations (19)) is the field given by equation (39), produced by half-space 2 in the region $z > -d/2$. All the quantities pertaining to half-spaces 1, 2 will get a suffix 1 or, respectively, 2. Introducing these fields in equations (19) and (40) we get the dispersion equations

$$\begin{aligned}
\tfrac{\kappa_1'-\kappa}{\kappa_1'+\kappa}\cdot\tfrac{\kappa_2'-\kappa}{\kappa_2'+\kappa}e^{2i\kappa d} &= 1\ ,\\
\tfrac{\kappa_1'-\kappa}{\kappa_1'+\kappa}\cdot\tfrac{\kappa_2'-\kappa}{\kappa_2'+\kappa}\cdot\tfrac{\kappa\kappa_1'-k^2}{\kappa\kappa_1'+k^2}\cdot\tfrac{\kappa\kappa_2'-k^2}{\kappa\kappa_2'+k^2}e^{2i\kappa d} &= 1\ .
\end{aligned} \tag{41}$$

The solutions of these equations give the eigenfrequencies of the two electromagnetically-coupled half-spaces. Since, according to equation (18),

$$(\kappa' \pm \kappa)(\kappa\kappa' \pm k^2) = \lambda^2(\varepsilon\kappa \pm \kappa') \ , \tag{42}$$

the second dispersion equation (41) can also be written as

$$\frac{\kappa_1' - \varepsilon_1\kappa}{\kappa_1' + \varepsilon_1\kappa} \cdot \frac{\kappa_2' - \varepsilon_2\kappa}{\kappa_2' + \varepsilon_2\kappa} e^{2i\kappa d} = 1 \ , \tag{43}$$

where $\varepsilon_{1,2}(\omega)$ are the dielectric functions of the two half-spaces. These dispersion equations have been established in Refs. [8], [10, 11], by using continuity conditions for the electromagnetic field at the surfaces of the two half-spaces.

6 Casimir force

In general, the dispersion equations (41) have not solutions. However, there exist particular conditions, corresponding precisely to physically interesting cases, which ensure solutions for the dispersion equations (41). For instance, conductors are characterized by $\omega_c = 0$ and large values of ω_p. In this case, the z-component κ' of the wavevector is purely imaginary and its magnitude acquires large values in comparison with κ (*i.e.*, λ). Purely imaginary wavevectors κ' correspond to damped surface plasmon-polariton modes in conductors (see, for instance, Refs. [38, 40]), in agreement with the original Casimir's assumption concerning the boundary conditions at the surfaces of two semi-infinite metals. In this retarded regime of interaction the electromagnetic field is propagating between the two half-spaces (κ real), but it is damped along the z-direction inside the conducting half-spaces. Good dielectrics are characterized by $\omega \ll \omega_c \ll \omega_p$, so that κ' (which is real) acquires again large values. This condition is usually referred to as the condition of long wavelengths in comparison with the main (characteristic) absorption wavelength of the substance (see, for instance, Ref. [15]). It is easy to see that equations (41) have solutions $\kappa d = \pi n$, n any integer, for $\left|\kappa_{1,2}'\right| \gg \kappa_{1,2}, \left|\varepsilon_{1,2}\right|\kappa_{1,2}$. Solutions $\kappa d = \pi n$ can be easily understood. In the in-between region there is a field produced by the half-space 1, which goes like $\mathbf{E}^{(1)}, \mathbf{H}^{(1)} \sim e^{-i\kappa z}$ and a field produced by the half-space 2, which goes like $\mathbf{E}^{(2)}, \mathbf{H}^{(2)} \sim e^{i\kappa z}$. Cross-terms of the form $\mathbf{E}^{(1)*}\mathbf{E}^{(2)}$, integrated over z from $-d/2$ to $d/2$, in the energy of the electromagnetic field in this region give rise to the factor $\sin\kappa d$. The condition $\kappa d = \pi n$ ensures the vanishing of this interaction energy. There is also an interaction electromagnetic energy inside the two half-spaces (involving cross-terms), which cannot, in general, be removed, except in those cases where it is practically negligible. This condition correspond to $\left|\kappa_{1,2}'\right| \gg \kappa_{1,2}, \left|\varepsilon_{1,2}\right|\kappa_{1,2}$.

The solutions $\kappa d = \pi n$ ($\kappa = \sqrt{\lambda^2 - k^2}$) imply the eigenfrequencies

$$\Omega_n(k) = c\sqrt{k^2 + \frac{\pi^2 n^2}{d^2}} \ ; \tag{44}$$

according to equations (19) and (40); the corresponding amplitudes can be written as

$$A_{1,2,n} = 2\pi a_{1,2,n}\delta(\omega - \Omega_n(k)) \ , \tag{45}$$

where $u_{1,2,n}(\mathbf{k}, z; t) = a_{1,2,n}e^{i\Omega_n(k)t}e^{i\kappa_{1,2}'z}$. We can see that $a_{1,2,n}$ are displacements, according to equation (7), and they correspond to the coordinates of harmonic-oscillators with frequencies $\Omega_n(k)$. According to equations (40), a similar representation holds for the amplitudes $B_{1,2}$ of the displacement field in the half-space 2, as well as for the associated electromagnetic fields. In effect, the coordinates of the $a_{1,2,n}$-type are the coordinates of the normal modes (labelled by $\mathbf{k}$ and n) of the two electromagnetically-coupled half-spaces. The energy asssociated with these normal modes, implied by equation (45) (or the similar equation for amplitudes $B_{1,2,n}$), is the total energy of the ensemble (*i.e.*, the matter energy and the energy of the electromagnetic field, of the form given in equations (29) and (30)) into the energy of an infinite set of harmonic oscillators with frequencies $\Omega_n(k)$; the energy of the electromagnetic field plays the role of interacting energy, while the polarization degrees of freedom of the two half-spaces are associated with the dynamical variables (matter energy). The motion of the normal modes can be quantized, according to standard rules, so that the ground-state energy is given by

$$E = \sum_{n=0}^{\infty} \sum_{\mathbf{k}} \hbar\Omega_n(k) = \frac{S\hbar c}{2\pi} \sum_{n=0} \int_0 dk \cdot k\sqrt{k^2 + \frac{\pi^2 n^2}{d^2}} \ , \tag{46}$$

where S denotes the area of the surface and factor 2 has been introduced in order to account for the two labels 1 and 2.

We estimate the change brought about by the finite distance d in the energy E by using the Euler-Maclaurin formula:[49]

$$\Delta E = \sum_{m=1} \frac{(-1)^m B_m (\pi/d)^{2m-1}}{(2m)!} f^{(2m-1)}(0) \ , \tag{47}$$

where B_m are the Bernoulli's numbers and

$$f(\kappa) = \frac{S\hbar c}{2\pi} \int dkk\sqrt{k^2 + \kappa^2} \ ; \tag{48}$$

introducing $u = k^2 + \kappa^2$, equation (47) becomes

$$\Delta E = \frac{\hbar c S}{4\pi} \sum_{m=1} \frac{(-1)^m B_m (\pi/d)^{2m-1}}{(2m)!} \left(\int_{\kappa^2} du\sqrt{u} \right)_0^{(2m-1)} \ , \tag{49}$$

The only contribution to equation (49) comes from the third-order derivative. We get ($B_2 = 1/30$)

$$\Delta E = -\frac{\pi^2 \hbar c S}{720} \cdot \frac{1}{d^3} \tag{50}$$

and an attractive force

$$F = -\frac{\pi^2 \hbar c S}{240} \cdot \frac{1}{d^4} \ , \tag{51}$$

which is the well-known Casimir force, acting between two half-spaces with parallel surfaces separated by distance d. We can see that it is the same for dielectrics and conductors (under the conditions given before), including the pair conductor-dielectric, does not depend on the nature of the two semi-infinite bodies and arises from the zero-point (vacuum) fluctuations of the motion of the charge carriers in the two polarizable bodies. We may say that it has a universal character.

The effect of the temperature $T = 1/\beta$ can be incorporated in equation (49) by the change

$$\int_{\kappa^2} du\sqrt{u} \rightarrow \int_{\kappa^2} du\sqrt{u} \coth\left[\frac{1}{2}\beta\hbar c\sqrt{u}\right] \ . \tag{52}$$

For realistic values of the parameters we have $\beta\hbar c/d \gg 1$, so we get a temperature correction factor $\simeq \coth(\beta\hbar c/d)$ in the expression of the force.

7 van der Waals-London force

For shorter distances d, the electromagnetic field acquires the non-retarded regime corresponding to $\lambda \to 0$; it follows that $\kappa \simeq ik$, *i.e.* the electromagnetic field is damped along the z -direction, both inside and ouside the half-spaces. In this limit we have

$$\kappa' \simeq \kappa - \frac{\lambda^2 \omega_p^2}{2\kappa(\omega^2 - \omega_c^2)} \ , \ \kappa\kappa' + k^2 \simeq \lambda^2 \left[1 - \frac{\omega_p^2}{2(\omega^2 - \omega_c^2)} \right] \tag{53}$$

and $\kappa\kappa' - k^2 \simeq -2k^2$. Making use of these approximations, the second equation (41) leads to

$$(\omega^2 - \omega_{c1}^2 - \frac{1}{2}\omega_{p1}^2)((\omega^2 - \omega_{c2}^2 - \frac{1}{2}\omega_{p2}^2) = \frac{1}{4}\omega_{p1}^2 \omega_{p2}^2 e^{-2kd} \ . \tag{54}$$

We solve this equation for large values of the kd, which bring the main contribution to integrals over $\mathbf{k}$. Within this approximation, the *rhs* of equation (54) may be treated as a small perturbation. From the zero-point energy, we get the van der Waals-London force (per unit area) for distinct bodies

$$F = -\frac{\hbar \omega_{p1} \omega_{p2}}{16\pi\sqrt{2} C_1 C_2 (\omega_{p1} C_1 + \omega_{p2} C_2)} \cdot \frac{1}{d^3} \ , \tag{55}$$

where

$$C_{1,2} = \sqrt{\frac{\varepsilon_{01,2} + 1}{\varepsilon_{01,2} - 1}} \ , \tag{56}$$

$\varepsilon_{01,2}$ being the static dielectric constants (for conductors, $C_{1,2} \to 1$). For identical bodies, the force becomes

$$F = -\frac{\hbar \omega_p}{32\pi\sqrt{2}} \left(\frac{\varepsilon_0 - 1}{\varepsilon_0 + 1} \right)^{3/2} \cdot \frac{1}{d^3} \tag{57}$$

(for conductors $|\varepsilon_0| \to \infty$).

8 A third body

We assume now that a slab of thickness d and parameters ω_{p3}, ω_{c3} (body 3) is inserted in the gap between the two half-spaces. All the calculations given in **Sections 2** and **3** are repeated for this body, which brings its own component κ_3' of the wavevevector along the z-axis, given by

$$\kappa_3'^2 = \kappa^2 - \frac{\lambda^2 \omega_{p3}^2}{\omega^2 - \omega_{c3}^2} = \varepsilon_3 \lambda^2 - k^2 \ , \tag{58}$$

ε_3 being the dielectric function of this body. The first dispersion equation (41) becomes now

$$\begin{gathered} \left(\frac{\kappa_1'+\kappa}{\kappa_1'-\kappa} \cdot \frac{1}{\kappa_3'+\kappa} - \frac{1}{\kappa_3'-\kappa} \right) \left(\frac{\kappa_2'+\kappa}{\kappa_2'-\kappa} \cdot \frac{1}{\kappa_3'+\kappa} - \frac{1}{\kappa_3'-\kappa} \right) e^{2i\kappa_3' d} = \\ \left(\frac{\kappa_1'+\kappa}{\kappa_1'-\kappa} \cdot \frac{1}{\kappa_3'-\kappa} - \frac{1}{\kappa_3'+\kappa} \right) \left(\frac{\kappa_2'+\kappa}{\kappa_2'-\kappa} \cdot \frac{1}{\kappa_3'-\kappa} - \frac{1}{\kappa_3'+\kappa} \right) \ , \end{gathered} \tag{59}$$

while the second dispersion equation (41) can be written as

$$(a_1 b_- - b_+)(a_2 b_- - b_+) e^{2i\kappa_3' d} = (a_1 b_+ - b_-)(a_2 b_+ - b_-) \ , \tag{60}$$

where

$$a_i = \frac{\kappa\kappa_i' + k^2}{\kappa\kappa_i' - k^2} \cdot \frac{\kappa_i' + \kappa}{\kappa_i' - \kappa} = \frac{\varepsilon_i\kappa + \kappa_i'}{\varepsilon_i\kappa - \kappa_i'} \ , \ i = 1, 2 \tag{61}$$

and

$$b_\pm = \frac{\kappa\kappa_3' \pm k^2}{\kappa_3' \mp \kappa} \ . \tag{62}$$

We can see that the dispersion equations (41) can be retrieved from equations (59) and (60) by putting formally $\kappa_3' = \kappa$, as for vacuum.

For large values of $\left|\kappa_{1,2}'\right|$ (either conductors or dielectrics), equations (59) and (60) have solution $\kappa_3' d = \pi n$, n integer, which implies $\varepsilon_3(\omega)\lambda^2 = c^2 K_3'^2$, where $\mathbf{K}_3' = (\mathbf{k}, \pi n/d)$. This equation has two solution branches, one starting at $\sqrt{\omega_{p3}^2 + \omega_{c3}^2}$ with an asymptote $\simeq cK_3'$, and another starting as vK_3' and asymptote ω_{c3}, where

$$v = c\frac{\omega_{c3}}{\sqrt{\omega_{p3}^2 + \omega_{c3}^2}} = \frac{c}{\sqrt{\varepsilon_{30}}} \ , \tag{63}$$

ε_{30} being the (static) dielectric constant of the body 3. These are the well-known polariton branches in a polarizable body. It follows that the Casimir force is given by the same equation (51) with the renormalized light velocity (polariton velocity) v, as expected. For a conducting body inserted in the gap (κ_3' purely imginary), the force is vanishing.

In the non-retarded regime $\kappa \simeq ik$ the situation is more complicated. Equation (60) leads to

$$\begin{gathered}\left[4(\omega^2 - D_1)(\omega^2 - D_3) - \omega_{p1}^2\omega_{p3}^2\right]\left[4(\omega^2 - D_2)(\omega^2 - D_3) - \omega_{p2}^2\omega_{p3}^2\right] = \\ = 4\left[\omega_{p1}^2(\omega^2 - D_3) - \omega_{p3}^2(\omega^2 - D_1)\right]\left[\omega_{p2}^2(\omega^2 - D_3) - \omega_{p3}^2(\omega^2 - D_2)\right] e^{-2kd} \ ,\end{gathered} \tag{64}$$

where

$$D_i = \frac{1}{2}\omega_{pi}^2\frac{\varepsilon_{0i} + 1}{\varepsilon_{0i} - 1} \ , \ i = 1, 2, 3. \tag{65}$$

The zero-point energy associated with the solutions of this equation leads to the van der Waals-London force. It is easy to see that for large values of D_3 (weak dielectric in-between), equation (64) becomes equation (54),which means that the effect of a weak dielectric introduced in the gap between the two half-spaces is a second-order correction. For two identical conductors 1 and 2 and a distinct conductor 3 in-between the force is given by

$$F = -\frac{\hbar}{32\pi\sqrt{2}}\frac{\omega_p^2 - \omega_{p3}^2}{(\omega_p^2 + \omega_{p3}^2)^{3/2}} \cdot \frac{1}{d^3} \ . \tag{66}$$

More complicated situations can be treated by solving equation (64).

9 Formulae of the theory of the electromagnetic fluctuations

We give here a formal deduction of the formulae obtained within the framework of the theory of the electromagnetic fluctuations, following Refs. [8], [10, 11]

Suppose that the eigenvalues $\Omega_n(\mathbf{k})$ are given by the roots of an equation written as $G(\omega, k) = 0$, like one of equations (41). Then, the zero-point energy can be written as

$$E = \frac{1}{2}\hbar\sum_{n\mathbf{k}}\Omega_n(k) = \frac{\hbar}{4\pi i}\sum_{n\mathbf{k}}\int d\omega\frac{\omega}{\omega - \Omega_n(k)} \ , \tag{67}$$

or

$$E = \frac{\hbar}{2i}\int dkk\int d\omega\omega\frac{\partial}{\partial\omega}\ln G \tag{68}$$

(per unit area), where the integration with respect to ω is performed around the positive ω-axis (we assume that function G has no poles). We pass from the variables $(\omega,\, k)$ to the variables $(\xi,\, p)$ defined by

$$\omega = i\xi\ ,\ p = \sqrt{1 + c^2k^2/\xi^2}sgn(\xi)\ . \tag{69}$$

The jacobian of this transformation is

$$\frac{\partial(\omega,\, k)}{\partial(\xi,\, p)} = \frac{i\xi p}{c(p^2-1)^{1/2}} \tag{70}$$

and the integration is represented as

$$\int_{-\infty}^{-1} dp\int_{-\infty}^{0} d\xi - \int_{1}^{\infty} dp\int_{0}^{\infty} d\xi \tag{71}$$

We take for $G = 0$ equations (41), which, with the new variables, become

$$\begin{aligned} G_1 &= \tfrac{(s_1+p)(s_2+p)}{(s_1-p)(s_2-p)}e^{2\xi pd/c} - 1 = 0\ , \\ G_2 &= \tfrac{(s_1+\varepsilon_1 p)(s_2+\varepsilon_2 p)}{(s_1-\varepsilon_1 p)(s_2-\varepsilon_2 p)}e^{2\xi pd/c} - 1 = 0\ , \end{aligned} \tag{72}$$

where $s_i = (\varepsilon_i - 1 + p^2)^{1/2}$, $i = 1, 2$ and κ is replaced by $\kappa = -i\xi p/c$. The derivative with respect to ω in equation (68) becomes

$$\frac{\partial G}{\partial\omega} = -i\frac{\partial G}{\partial\xi} + i\frac{p^2-1}{p\xi}\frac{\partial G}{\partial p}\ . \tag{73}$$

In order to get the force, we take the (minus) derivative with respect to d in equation (68) and make use of

$$\frac{\partial G}{\partial d} = \frac{2\xi p}{c}(G+1)\ . \tag{74}$$

Combining equations (73) and (74), we get easily

$$\frac{\partial}{\partial d}\left(\frac{1}{G}\frac{\partial G}{\partial\omega}\right) = \frac{2}{ic}\left(\frac{1}{p} + \frac{1}{pG} - \frac{\xi p}{G^2}\frac{\partial G}{\partial\xi} + \frac{p^2-1}{G^2}\frac{\partial G}{\partial p}\right)\ . \tag{75}$$

An integration by parts in $F = \partial E/\partial d$ leads to the force

$$F = -\frac{\hbar}{2\pi^2c^3}\int_1^\infty dpp^2\int_0^\infty d\xi\xi^3\left(\frac{1}{G_1} + \frac{1}{G_2}\right)\ , \tag{76}$$

which is the well-known formula given in Refs. [13]-[17]. For finite temperatures the integration over ξ is replaced by a summation over the integers n, such as $\beta\hbar\xi_n = 2\pi n$, where $\beta = 1/T$ is the reciprocal of the temperature T.

For conductors, in the non-retarded limit, equation (76) leads to the Casimir force given by equation (51). For poor dielectrics, or combinations of poor dielectrics with conductors, equation (76) brings a small correction factor in the Casimir force (see, for instance, equation (82.6) in Ref. [15]), which indicates, in fact, that the force is vanishing in this case. In the limit of good dielectrics, equation (76) leads to the same universal Casimir force given by equation (51).

In the non-retarded limit $\omega \rightarrow 0$ ($\xi \rightarrow 0$), the most important contribution to the p-integral in equation (76) comes from $p \gg 1$, due to the presence of the exponential in the denominator. Consequently, we may take $s_{1,2} \simeq p$, which leads to

$$F \simeq -\frac{\hbar}{16\pi^2 d^3} \int_0^\infty dx x^2 \int_0^\infty d\xi \left[\frac{(1+\varepsilon_1)(1+\varepsilon_2)}{(1-\varepsilon_1)(1-\varepsilon_2)} e^x - 1 \right]^{-1} , \tag{77}$$

which is the well-known formula given in Refs. [13]-[17] for the van der Waals-London force. The evaluation of the ξ-integral is difficult, so we cannot compare the result with equation (55).

Both equations (76) and (77) can be extended to very rarefied bodies, leading to well-known forces computed quantum-mechanicaly for two interacting atoms (molecules).[15] In general, equations (76) and (77) are valid where there exist solutions of equation $G(\omega, k) = 0$ (equations (41)). Unfortunately, equations (76) and (77) may also indicate false solutions (as for poor dielectrics).

10 Concluding remarks. Sphere and half-space

Let us denote by $F_{1/2} = CS/d^n$ the van der Waals-London or Casimir force acting between two half-space separated by distance d, where C is a constant, S is the transverse area of the two half-spaces, $n = 3$ for the van der Waals-London force and $n = 4$ for the Casimir force. We look for a force $df = C_1/ |z|^{n_1} dV$, acting between the half-space and a "macroscopically infinitesimal" element of volume dV placed at distance $|z|$ from the half-space, such as

$$\int df = F_{1/2} , \tag{78}$$

where the integration is performed over the other half-space. We find easily $C_1 = Cn$ and $n_1 = n + 1$. Now we compute the force

$$F_s = \int df = Cn \int dV \frac{1}{(R + d - r\cos\theta)^{n+1}} \tag{79}$$

acting between the half-space and a sphere of radius R placed at distance d from the half-space (the distance between the half-space and the surface of the sphere); the integration in equation (79) is performed over the volume of the sphere. The integration in equation (79) is elementary, and, for $R \gg d$, we get the force

$$F_s \simeq \frac{2\pi CR}{(n-1)d^{n-1}} . \tag{80}$$

The force acting between a half-space and a spherical sell of radius R is $2\pi CR^2/d^n$. In a similar way we can derive the force acting between two bodies of any shape. The force acting between two macroscopic particles is given by

$$f = \frac{n(n+1)(n+2)C}{2\pi d^{n+4}} v_1 v_2 , \tag{81}$$

where $v_{1,2}$ are the volumes of the two particles.

Finally, it is worth commenting here on the meaning of the zero-point energy, as obtained by the present method.

The coupled equations of motion have been solved here both for the polarization (displacement) field (Newton's equation (1)) and the electromagnetic field obeying Maxwell's equations (equations (4)). It was found that the solution depends on two free parameters, $A_{1,2}$, which obey harmonic-oscillator equations of motion, with certain characteristic frequencies $\Omega_n(k)$ (equation (45)). Consequently, the harmonic-oscillator hamiltonian was set up for these dynamical variables and its zero-point energy has been calculated. On the other hand, inserting the solution of the equations of motion, containing the two free parameters $A_{1,2}$, into the matter and electromagnetic field energy (equations of the type (29) and (30)), we get an energy depending on these two parameters $A_{1,2}$. An expansion in normal modes shows that this energy is the same as the energy given by the clasical limit of the harmonic-oscillator hamiltonian.

Indeed, the problem can be cast into another form. We can start with a particles-in-the field hamiltonian of the form

$$H_m = q\Phi + \frac{1}{2m}(\mathbf{p} - \frac{q}{c}\mathbf{A})^2 \tag{82}$$

for matter interacting with the electromagnetic field in bodies 1 and 2 (with corresponding integration over volume in equation (82)). The electromagnetic energy of the form

$$E_{em} = \frac{1}{8\pi}\int d\mathbf{R}(|\mathbf{E}|^2 + |\mathbf{H}|^2) \tag{83}$$

must be add to the energy given by equation (82), for each bodies 1 and 2 as well as for the in-between space (for the third body a corresponding energy, both of matter and the electromagnetic field should be added). It is well known that an energy of the form given by equation (83) can be written as a sum of energies of independent oscillators (including the longitudinal and the scalar deegres of freedom of the electromagnetic field);[50] it reads

$$H_{em} = \frac{1}{2}\sum_r (|p_r|^2 + \omega_r^2 |q_r|^2) \ , \tag{84}$$

where q_r are the normal-modes coordinates and p_r denote their momentum. Equation (84) can be viewed as the hamiltonian of the electromagnetic field. It is obtained from equation (83) by introducing the normal coordinates for the vector and scalar potentials of the form

$$\mathbf{A} = \sum_r q_r(t)\mathbf{A}_r(\mathbf{R}) \ , \tag{85}$$

where

$$\Delta\mathbf{A}_r + \frac{\omega_r^2}{c^2}\mathbf{A}_r = 0 \ , \ \omega_r = cK_r \ , \tag{86}$$

the wavevector $\mathbf{K}_r$ being given by the boundary conditions. We can check then the Maxwell equations for potentials, by $\ddot{q}_r + \omega_r^2 q_r = 0$.

Therefore, we have matter hamiltonians (equation (82)) and electromagnetic hamiltonians (equation (84)) for each of the three regions, the interaction between matter and the electromagnetic field being included in the matter hamiltonians. At first sight, there is no interaction between the two bodies (or the three space regions). However, we should add the continuity equations (or discontinity conditions) for the electromagnetic field at the surfaces of separation of the three space regions. This is usually done by introducing the dielectric functions of the material (non-magnetic) bodies, which amounts to removing the explicit effect of the interaction of matter with the electromagnetic field. There are cases where the dielectric funtion is used together with preserving the interaction in the matter hamiltonian, which is an inconsistemt treatment.[13]-[17] It amounts to mistake an external field (charge, current) for the internal polarization. We are then left with electromagnetic fields for three space regions, which are not independent. From all the electromagnetic hamiltonians of the form given by equation (84) we are left with one, for two types of degrees of freedom q_r, which correspond precisely to a harmonic-oscillator hamiltonian for the free parameters $A_{1,2}$. As a mater of fact, the treatment given in Refs. [8], [10, 11] is very close to the one sketched here.

The force acting between the two bodies is derived from the change in the zero-point energy with the distance between the two bodies. It is worth noting that it is a quantum effect, which is vanishing in the classical limit. Very often, the stress tensor of the electromagnetic field (energy flux through the separation surface) is used to derive the force.[13]-[17] In the classical limit this energy flux is not vanishing, except for finite-size bodies. For finite-size bodies, it is vanishing also in the quantum limit. It folows that the molecular forces do not originate in the Lorentz force, nor in the variation of the electromagnetic momentum (but in the vacuum fluctuations). A corresponding stress tensor (energy flux) can be determined from this force, which is the opposite of soem of the usual treatments.

Acknowledgments. The author is indebted to the members of the Laboratory of Theoretical Physics and Condensed Matter in the Institute for Physics and Nuclear Engineering at Magurele-Bucharest for many useful discussions. The author is also indebted to the members of the Seminar of the Institute for Atomic Physics, Magurele-Bucharest for a thorough analysis of the results presented here.

References

[1] R. Eisenschitz and F. London, "Uber das Verhaltnis der van der Waalsschen Krafte zu den homoopolaren Bindungskraften", Z. Physik **60** 491-527 (1930).

[2] F. London, "Zur Theorie und Systematik der Molekularkrafte", Z. Physik **63** 245-279 (1930).

[3] F. London, "The general theory of molecular forces", Trans. Faraday Soc. **33** 8-26 (1937).

[4] H. B. G. Casimir and D. Polder, "The influence of retardation on the London-van der Waals forces," Phys. Rev. **73** 360-372 (1948).

[5] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 3 (*Quantum Mechanics*), Butterworth-Heinemann, Oxford (2003), p. 341.

[6] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 4 (*Quantum Electrodynamics*), Butterworth-Heinemann, Oxford (2003), p. 347.

[7] H. Casimir, "On the atttraction between two perfectly conducting plates," Proc. Kon. Ned. Ak. Wet. **51** 793 (1948).

[8] N. G. Van Kampen, B. R. A. Nijboer and K. Schram, "On the macroscopic theory of van der Waals forces," Phys. Lett. **A26** 307-308 (1968).

[9] T. H. Boyer, "Recalculations of long-range van der Waals potentials", Phys. Rev. **180** 19-24 (1969).

[10] E. Gerlach, "Equivalence of van der Waals forces between solids and the surface-plasmon interaction," Phys. Rev. **B4** 393-396 (1971).

[11] K. Schram, "On the macroscopic theory of retarded van der Waals forces," Phys. Lett. **A43** 282-284 (1973).

[12] J. Heinrichs, "Theory of van der Waals interaction between metal surfaces," Phys. Rev. **B11** 3625-3636 (1975).

[13] E. Lifshitz, ZhETF **29** 94-105 (1956) (Sov. Phys. JETP **2** 73-83 (1956)).

[14] I. E. Dzyaloshinskii, E. M. Lifshitz and L. P. Pitaevskii, "The general theory of van der Waals forces," Adv. Phys. **10** 165-209 (1961).

[15] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 5 (*Statistical Physics*), part. 2, Butterworth-Heinemann, Oxford (2003), ch. 8.

[16] J. Schwinger, "Casimir effect in source theory", Lett. Math. Phys. **1** 43-47 (1975).

[17] J. Schwinger, L. L. deRaad, Jr., and K. A. Milton, "Casimir effect in dielectrics", Ann. Phys. **115** 1-23 (1978).

[18] P. W. Milloni, *The Quantum Vacuum* (Academic Press, San Diego, 1994).

[19] V. M. Mostepanenko and N. N. Trunov, *The Casimir Effect and its Applications* (Clarendon, Oxford, 1997).

[20] S. K. Lamoreaux, "Demonstration of the Casimir force in the 0.6 to 6μm range," Phys. Rev. Lett. **78** 5-8 (1997).

[21] L. H. Ford, "Casimir sphere between a dielectric sphere and a wall: a model for amplification of vacuum fluctuations, Phys. Rev. **A58** 4279-4286 (1998).

[22] U. Mohideen and A. Roy, "Precision measurement of the Casimir force from 0.1 to 0.9μm", Phys. Rev. Lett. **81** 4549-4552 (1998).

[23] S. K. Lamoreaux, "Calculation of the Casimir force between imperfectly conductiong plates," Phys. Rev. **A59** R3149-R3153 (1999).

[24] A. Lambrecht and S. Reynard, "Comment on "Demonstration of the Casimir force in the 0.6 to 6μm range"," Phys. Rev. Lett. **84** 5672-5672 (2000).

[25] A. Lambrecht and S. Reynaud, "Casimir force between metallic mirrors", Eur. Phys. J. **D8** 309-318 (2000).

[26] M. Bordag, U. Mohideen and V. M. Mostepanenko, "New developments in the Casimir effect," Phys. Reps. **353** 1-205 (2001).

[27] K. A. Milton, *The Casimir Effect* (World Scientific, Singapore, 2001).

[28] G. Bressi, G. Carugno, R. Onofrio and G. Ruoso, "Measurement of the Casimir force between parallel metallic surfaces", Phys. Rev. Lett. **88** 041804 (2002) (1-4).

[29] C. Genet, A. Lambrecht and S. Reynaud, "Casimir force and the quantum theory of lossy optical cavities," Phys. Rev. **A67** 043811 (1-18) (2003).

[30] K. A. Milton, "The Casimir effect: recent controversies and progress", J. Phys. A: Math. Gen. **37** R209-R277 (2004).

[31] F. Chen, U. Mohideen, G. L. Klimchitskaya and V. M. Mostepanenko, "Investigation of the Casimir force between metal and semiconductor test bodies," Phys. Rev. **A72** 020101(R1-4) (2005).

[32] S. K. Lamoreaux, "The Casimir force: background, experiments and applications," Reps. Progr. Phys. **65** 201-236 (2005).

[33] F. Intravaia, "Effet Casimir et interaction entre plasmons de surface," These de Doctorat de l'Universite Paris VI, 1-177, June 2005.

[34] J. M. Obrecht, R. J. Wild, M. Antezza, L. P. Pitaevskii, S. Stringari and E. A. Cornell, "Measurement of the temperature dependence of the Casimir-Polder force," Phys. Rev. Lett. **98** 063201 (1-4) (2007).

[35] F. Intravaia, C. Henkel and A. Lambrecht, "Role of surface plasmons in the Casimir effect," Phys. Rev. **A76** 033820 (1-11) (2007).

[36] T. Emig, N. Graham, R. L. Jaffe and M. Kardar, "Casimir forces between arbitrary compact objects", Phys. Rev. Lett. **99** 170403 (2007) (1-4).

[37] M. Kruger, T. Emig, G. Bimonte and M. Kardar, "Non-equilibrium Casimir forces:: spheres and sphere-plate", Eur. Phys. Lett. **95** 21002 (2011) (1-6).

[38] M. Apostol and G. Vaman, "Electromagnetic eigenmodes in matter. van der Waals-London and Casimir forces", Pogr. Electrom. Res. PIER **B19** 115-131 (2010).

[39] M. Apostol and G. Vaman, "Attraction force between a polarizable point-like particle and a semi-infinite solid", Roum. J. Phys. **55** 764-771 (2010).

[40] M. Apostol and G. Vaman, "Electromagnetic field interacting with a semi-infinite plasma", J. Opt. Soc. Am. **A26** 1747-1753 (2009).

[41] M. Apostol and G. Vaman, "Plasmons and difraction of an electromagnetic plane wave by a metallic sphere", Progr. Electrom. Res. PIER **98** 97-118 (2009).

[42] P. Drude, "Zur Elektronentheorie der Metalle", Ann. Phys. **306** 566-613 (1900).

[43] P. Drude, "Zur Elektronentheorie der Metalle, 2. Teile. Galvanomagnetische und thermomagnetische Effecte, **308** 369-402 (1900).

[44] H. A. Lorentz, *The Theory of Electrons*, Teubner, Leipzig (1916).

[45] C. Kittel, *Introduction to Solid State Physics*, Wiley, NJ (2005).

[46] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, eds. A. Jeffrey and D. Zwillinger, 6th edition, Academic Press, San Diego (2000), pp. 714-715, 6.677.1,2.

[47] M. Born and E. Wolf, *Principles of Optics*, Pergamon, London (1959).

[48] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 2 (*The Classical Theory of Fields*), Butterworth-Heinemann, Oxford (2003).

[49] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Mathematical Library, Cambridge (2004), p. 128.

[50] W. Heitler, *The Quantum Tehroy of Radiation*, Dover, NY (1954).