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Antennas and Fields

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Point dipole. Consider a time-dependent displacement $u(t)$ along the z-axis, of a mobile charge q located in a body of a small volume v attached to the origin. The displacement vector can be written as

$$\mathbf{u} = v(0,\, 0,\, u(t))\delta(\mathbf{r}) \ , \tag{1}$$

the charge and current densities are

$$\rho = -(q/v)div\mathbf{u} = -qu(t)\frac{\partial}{\partial z}\delta(\mathbf{r}) \ , \ \mathbf{j} = (q/v)\dot{\mathbf{u}} = q(0,\, 0,\, \dot{u}(t))\delta(\mathbf{r}) \tag{2}$$

(continuity $\partial\rho/\partial t + div\mathbf{j} = 0$) and the polarization (dipole per unit volume) can be written as

$$\mathbf{P} = \frac{q}{v}\mathbf{u} = q(0,\, 0,\, u(t))\delta(\mathbf{r}) = (0,\, 0,\, p(t))\delta(\mathbf{r}) \ , \tag{3}$$

where $p(t) = qu(t)$ is the point dipole;

$$\rho = -div\mathbf{P} = -p(t)\frac{\partial}{\partial z}\delta(\mathbf{r}) \ , \ \mathbf{j} = \dot{\mathbf{P}} = (0,\, 0,\, \dot{p}(t))\delta(\mathbf{r}) \ . \tag{4}$$

Such a point-dipole approximation is valid for distances much larger than the dimensions of the body.

The Kirchhoff's (retarded) potentials are given by

$$\mathbf{A}(t,\mathbf{r}) = \frac{1}{c}\int d\mathbf{r}'\frac{\mathbf{j}(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} \ , \ \Phi(t,\mathbf{r}) = \int d\mathbf{r}'\frac{\rho(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} \ ; \tag{5}$$

they satisfy the Lorenz gauge $div\mathbf{A} + (1/c)\partial\Phi/\partial t = 0$, where c denotes the speed of light.

We take the temporal Fourier transform and compute the vector potential with the current given by equation (4):

$$A(\omega,\mathbf{r}) = A_z(\omega,\mathbf{r}) = -i\lambda p(\omega)\frac{e^{i\lambda r}}{r} \ , \tag{6}$$

where $\lambda = \omega/c$. The scalar potential can be obtained either by equation (5) or from the Lorenz gauge:

$$\Phi(\omega,\mathbf{r}) = -p(\omega)\frac{z}{r}\frac{\partial}{\partial r}\frac{e^{i\lambda r}}{r} \ . \tag{7}$$

these potentials are spherical waves (and their derivatives).

The electric field is given by $\mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t - grad\Phi$ and the magnetic field is given by $\mathbf{H} = curl\mathbf{A}$.

The near-field is obtained for $\lambda r \ll 1$, *i.e.* for distances much smaller than the wavelength (but much larger than the dimensions of the body); it follows that it is applicable for wavelengths much larger than the dimensions of the body. In this case we may neglect λ, and get

$$E_x = 3p\frac{xz}{r^5} \ , \ E_y = 3p\frac{yz}{r^5} \ , \ E_z = p(\frac{3z^2}{r^2} - 1)\frac{1}{r^3} \ , \ \mathbf{H} \simeq 0 \tag{8}$$

(where we omit the argument ω). The electric field has cylindrical symmetry, $E_\rho = 3pz/r^4$, $E_\theta = 0$ ($r^2 = \rho^2 + z^2$). It can be written in the more familiar form

$$\mathbf{E} = \frac{3(\mathbf{pr})\mathbf{r} - \mathbf{p}r^2}{r^5} \ , \tag{9}$$

where $\mathbf{p} = (0, 0, p)$; this is the quasi-static field of a dipole, in the near-field zone (stationary field, sub-wavelength zone). We note the longitudinal component ($\sim \mathbf{r}$) of the electric field.

In the wave zone $\lambda r \gg 1$ (distances much larger than the wavelengths) we retain only derivatives which imply λ:

$$\begin{array}{c} \mathbf{E} = -\lambda^2 p(\frac{xz}{r^2}, \frac{yz}{r^2}, \frac{z^2}{r^2} - 1)\frac{e^{i\lambda r}}{r} = \frac{\mathbf{H}\times\mathbf{r}}{r} \ , \\ \mathbf{H} = \lambda^2 p(\frac{y}{r}, -\frac{x}{r}, 0)\frac{e^{i\lambda r}}{r} = \lambda^2 \frac{\mathbf{r}\times\mathbf{p}}{r}\frac{e^{i\lambda r}}{r} \ . \end{array} \tag{10}$$

We can see that $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{r}$ are orthogonal to each other, *i.e.* the electromagnetic field is transverse in the wave zone; locally, the spherical wave propagates along the wavevector $\mathbf{k}$ directed along $\mathbf{r}$, $k = \lambda$, *i.e.* it is a plane wave. The Poynting vector is

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H} = \frac{c}{4\pi}H^2\frac{\mathbf{r}}{r} \ ; \tag{11}$$

its magnitude gives the energy radiated per unit time and per unit cross-sectional area.

In equation (11) the magnetic field is $\mathbf{H}(t)$, and it is real. The inverse Fourier transform in equation (10) gives the retarded field

$$\mathbf{H}(t) = \left.\frac{\ddot{\mathbf{p}} \times \mathbf{r}}{c^2 r^2}\right|_{t-r/c} \ , \tag{12}$$

such that

$$\mathbf{S} = \left.\frac{(\ddot{\mathbf{p}} \times \mathbf{r})^2}{4\pi c^3 r^4}\right| \frac{\mathbf{r}}{r}_{t-r/c} \tag{13}$$

and its magnitude

$$S = \left.\frac{\ddot{p}^2}{4\pi c^3 r^2}\right|_{t-r/c} \sin^2\theta \ , \tag{14}$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{r}$; therefore, the total energy radiated per unit time (the power) is

$$dE/dt = \left.\frac{2}{3c^3}\ddot{p}^2\right|_{t-r/c} \ . \tag{15}$$

For a periodic motion $p = p_0 \cos\Omega t$, we have

$$S = \frac{\Omega^4 p_0^2}{4\pi c^3 r^2}\sin^2\theta\cos^2\Omega t \ , \ dE/dt = \frac{2\Omega^4 p_0^2}{3c^3}\cos^2\Omega t \ ; \tag{16}$$

or, the time average

$$\overline{S}=\frac{\Omega^4p_0^2}{8\pi c^3r^2}\sin^2\theta \ , \ \overline{dE/dt}=\frac{\Omega^4p_0^2}{3c^3} \ . \tag{17}$$

The same result can be obtained by using the time average over a long time T for the Fourier transforms:

$$\begin{gathered}\overline{\mathbf{S}}=\frac{1}{T}\int dt\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}=\frac{1}{2\pi T}\int d\omega\frac{c}{4\pi}\mathbf{E}(-\omega)\times\mathbf{H}(\omega)=\\ =\frac{1}{2\pi T}\int d\omega\frac{c}{4\pi}\left|\mathbf{H}(\omega)\right|^2\frac{\mathbf{r}}{r}=\frac{1}{2\pi T}\int d\omega\frac{\omega^4}{4\pi c^3r^2}\left|p(\omega)\right|^2\sin^2\theta\frac{\mathbf{r}}{r} \ .\end{gathered} \tag{18}$$

Here, we may introduce $p(\omega)=\pi p_0[\delta(\omega-\Omega)+\delta(\omega+\Omega)]$ and use $\delta(\omega=0)=T/2\pi$.

$$\overline{\mathbf{S}}=\frac{\Omega^4p_0^2}{8\pi c^3r^2}\sin^2\theta\frac{\mathbf{r}}{r} \ , \tag{19}$$

which is equation (17).

Polarizability. Inside the body, the displacement $\mathbf{u}$ of the charges q with mass m and concentration n are subjected to Newton's equation of motion

$$m\ddot{\mathbf{u}}=q(\mathbf{E}_{ext}+\mathbf{E})-m\omega_c^2\mathbf{u}-m\gamma\dot{\mathbf{u}} \ , \tag{20}$$

where $\mathbf{E}_{ext}$ is the external electric field and $\mathbf{E}$ is the internal (polarization) electric field, generated by the displacement $\mathbf{u}$, according to the potentials given by equations (5); ω_c is a local characteristic frequency and γ is a damping coefficient. Such an equation of motion can be of sufficient generality to describe the classical polarization of matter.

First we note that it is the non-relativistic Newton equation of motion, since the mobile charges in matter have a small velocity. Second, we note that it lacks the Lorentz force. The Lorentz force arising from an external magnetic field can be added, but the Lorentz force arising from the internal magnetic field would imply a quadratic contribution in the magnitude of the displacement $\mathbf{u}$ ($\mathbf{u}$ and the velocity $\dot{\mathbf{u}}$), which may be neglected: $\mathbf{u}$, which gives the polarization charges and currents ($\rho=-nqdiv\mathbf{u}$, $\mathbf{j}=nq\dot{\mathbf{u}}$), is considered small in comparison with its spatial variations. In general, we leave aside the matter magnetization (which contributes additional currents). We note also that that we leave aside the spatial dispersion in equation (20) (parameters ω_c, γ do not depend on position). Equation (20) can be generalized by including several displacements and characteristic frequencies, coupled motion modes and anisotropies, etc.

The polarization is $\mathbf{P}=nq\mathbf{u}$ and $\mathbf{E}_t=\mathbf{E}_{ext}+\mathbf{E}$; taking the temporal Fourier transform and according to its definition ($\mathbf{P}(\omega)=\chi(\omega)\mathbf{E}_t(\omega)$), we get from equation (20) the electric susceptibility

$$\chi(\omega)=-\frac{\omega_p^2}{4\pi}\frac{1}{\omega^2-\omega_c^2+i\omega\gamma} \tag{21}$$

(and the dielectric function $\varepsilon(\omega)=1+4\pi\chi(\omega)$), where $\omega_p=(4\pi nq^2/m)^{1/2}$ is the plasma frequency. Similarly,

$$\mathbf{j}(\omega)=-i\omega\mathbf{P}(\omega)=-i\omega\chi(\omega)\mathbf{E}_t(\omega)=\sigma(\omega)\mathbf{E}_t(\omega) \ , \tag{22}$$

where

$$\sigma(\omega)=-i\omega\chi(\omega)=\frac{\omega_p^2}{4\pi}\frac{i\omega}{\omega^2-\omega_c^2+i\omega\gamma} \tag{23}$$

is the conductivity. This is known as the Lorentz-Drude (plasma) model of matter polarization.

For $\omega_c = 0$ and low frequencies ($\omega \rightarrow 0$), we get from (23) the static conductivity $\sigma = \omega_p^2/4\pi\gamma$; the same result is obtained directky from the equation of motion (20), assuming a relaxation time τ, such as

$$m\ddot{\mathbf{u}} \simeq m\frac{\dot{\mathbf{u}}}{\tau} = q\mathbf{E}_t \ , \tag{24}$$

whence $\mathbf{j} = nq\dot{\mathbf{u}} = (nq^2\tau/m)\mathbf{E}_t$ and $\sigma = \omega_p^2\tau/4\pi$. It follows that $\gamma = 1/\tau$ is associated with the energy loss and $\omega_c = 0$ corresponds to conductors; in dielectrics $\omega_c \neq 0$.

In typical metals the velocity of the charge carriers is $\simeq 10^5 m/s$ and their mean free path (at room temperature) is $\simeq 10^3$Å; therefore $\tau = 10^3$Å$/10^5 m/s = 10^{-12}s$ and $\gamma \simeq 10^{12}s^{-1}$ ($1THz$). In typical dielectrics (semiconductors) the mean velocity of the charge carriers is $\simeq 10^4 m/s$ and the mean free path (at room temperature) is 10^2Å, such that γ is of the same order as in conductors; but the typical characteristic frequencies are much higher ($10^{13} - 10^{15}s^{-1}$; typical plasma frequency is $\simeq 10^{15}s^{-1}$). In the static limit the damping is dissipative in conductors; for higher frequencies, both in conductors and dielectrics, the damping becomes more and more reactive. Various combinations of susceptibilities and conductivities can be formed, for various models of polarizable matter.

For dimensions of the body much smaller than the relevant wavelengths we may omit the internal electric field in the equation of motion (20). Then, we get the polarizability $\alpha(\omega) = \chi(\omega)$, by its definition $\mathbf{P}(\omega) = \alpha(\omega)\mathbf{E}_{ext}(\omega)$. For small frequencies (quasi-static limit) and finite dimensions (surface effects) we can check from equations (5) that the internal (polarization) electric field is proportional to the polarization, $\mathbf{E} = -4\pi n\mathbf{P}$, where n is called the (de)polarization factor ($n = 1/3$ for a sphere). Then, we get $\mathbf{P} = \chi(\mathbf{E}_{ext} - 4\pi n\mathbf{P})$ and

$$4\pi\alpha = \frac{\varepsilon - 1}{n\varepsilon + 1 - n} \ ; \tag{25}$$

which is known as the Clausius-Mossotti (or Lorentz-Lorenz) relation. Making use of the susceptibility (dielectric function) given by equation (21) for low frequencies, we get

$$\alpha = -\frac{\omega_p^2}{4\pi}\frac{1}{\omega^2 - \omega_c^2 - n\omega_p^2 + i\omega\gamma} \ , \tag{26}$$

which changes appreciably the susceptibility (for small dimensions), especially for conductors.

For a body of a small volume v, subjected to an external field $\mathbf{E}_{ext}$ (at the location of the body), we may write the polarization as

$$\mathbf{P}(\omega) = \alpha(\omega)v\mathbf{E}_{ext}(\omega)\delta(\mathbf{r}) \ , \tag{27}$$

so that it can be aproximated by a point dipole; its charge and current densities are

$$\rho = -\alpha(\omega)v[\mathbf{E}_{ext}(\omega)grad]\delta(\mathbf{r}) \ , \ \mathbf{j} = -i\omega\alpha(\omega)v\mathbf{E}_{ext}(\omega)\delta(\mathbf{r}) \ . \tag{28}$$

The formulae derived above for the point dipole can be applied, with $\mathbf{p}(\omega) = \alpha(\omega)v\mathbf{E}_{ext}(\omega)$. We can see that such a polarizable body subjected to the action of an external electric field radiates energy. The average Poynting vector is

$$\overline{\mathbf{S}} = \frac{v^2}{8\pi^2c^3Tr^2}\int d\omega\omega^4 \left|\alpha(\omega)\mathbf{E}_{ext}(\omega)\right|^2 \sin^2\theta\frac{\mathbf{r}}{r} \ , \tag{29}$$

where θ is the angle between $\mathbf{E}_{ext}$ and $\mathbf{r}$; the average energy radiated per unit time is given by

$$\overline{dE/dt} = \frac{v^2}{3\pi c^3T}\int d\omega\omega^4 \left|\alpha(\omega)\mathbf{E}_{ext}(\omega)\right|^2 \ . \tag{30}$$

Making use of $\alpha(\omega) = \chi(\omega)$ and $\chi(\omega)$ given by equation (21) we can estimate the integral in equation (30). By straightforward calculations we get

$$\overline{dE/dt} = \frac{v^2}{3c^3(4\pi)^2\gamma T}\omega_p^4\omega_c^2 \left|\mathbf{E}_{ext}(\omega_c)\right|^2 \ . \tag{31}$$

The average over time T implies an infrared cutoff frequency of the order of γ, such that we may take $\gamma T = 1$ (this can be checked directly in the limit $\gamma \rightarrow 0$ by using the representation of the polarizability by means of the principal value and the δ-function).

For a monochromatic external field $\mathbf{E}_{ext}(\omega) = \pi\mathbf{E}_{ext}[\delta(\omega - \Omega) + \delta(\omega + \Omega)]$ we get

$$\overline{dE/dt} = \frac{v^2}{3c^3}\Omega^4 \left|\alpha(\Omega)\right|^2 \left|\mathbf{E}_{ext}\right|^2 \ . \tag{32}$$

We can have an estimation of the fraction of radiated energy as $\simeq v\Omega^3 \left|\alpha(\Omega)\right|^2 / c^3$. This is a small coefficient, except for the resonance $\Omega \simeq \omega_c$, where it may become very large (similarly, the coefficient $v\omega_p^4/c^3\omega_c$ in equation (31) - which implies already a resonance - may be very large).

The above formulae are valid for $\omega_c \neq 0$. For $\omega_c = 0$ (conductors) we get, from equation (30),

$$\overline{dE/dt} = \frac{v^2}{3\pi c^3 T} \cdot \frac{\omega_p^4}{(4\pi)^2} \int d\omega \left|\mathbf{E}_{ext}(\omega)\right|^2 \ \ , n \tag{33}$$

where we can recognize the average electromagnetic energy fed into the body. For a monochromatic external field we get

$$\overline{dE/dt} = \frac{v^2}{3c^3} \cdot \frac{\omega_p^4}{(4\pi)^2} \left|\mathbf{E}_{ext}\right|^2 \ . \tag{34}$$

For extended bodies the situation is more complicated. In bodies of indefinite extension (inside the matter) we can identify eigenmodes of coupled charges and fields, called plasmons and polaritons. They acquire corresponding forms in bodies with surfaces, or under various other external conditions (an applied magnetic field, for instance). For dimensions of the bodies much larger than the relevant wavelengths we can define propagation rays, and we have reflection, refraction, transmission, etc. The energy fed into the bodies from the outside is radiated in the reflected, refracted or transmitted waves. For bodies of dimensions smaller than the wavelength (in the near-field zone, where we have also longitudinal components of the field), the approximation $\alpha = \chi$ (with possible inclusion of the depolarzing factors) is satsifactory; at large distance such bodies can be approximated by a point dipole; they radiate energy. For bodies situated in the near-field zone, with one or two dimensions infinite (and the others two or the other one small) the situation is complicated. We may imagine a longitudinal field "propagating" inside a slab (or a wire) of small thickness.

Infinite linear antenna and a panel of antennas. For an infinite linear antenna of thickness a we take the displacement field as

$$\mathbf{u}(t, \mathbf{r}) = a^2(0,\, 0,\, u(t))\delta(\mathbf{r}) \ , \tag{35}$$

where $\mathbf{r}$ is the transverse coordinate ($\mathbf{R} = (\mathbf{r}, z)$); the displacement does not depend on z, which means that actually we limit ourselves to very long wavelengths. We can see that the charge density is vanishing ($\rho = -nqdiv\mathbf{u}$, where q is the charge and n is the density of the mobile charges); we are left with the current density

$$\mathbf{j}(t, \mathbf{r}) = nqa^2(0,\, 0,\, \dot{u}(t))\delta(\mathbf{r}) = (0,\, 0,\, \dot{p}_l(t))\delta(\mathbf{r}) \tag{36}$$

($div\mathbf{j}=0$), where $p_l = nqa^2u$ is the linear density of dipoles. The vector potential is given by

$$A(\omega, \mathbf{r}) = A_z(\omega, \mathbf{r}) = -i\lambda p_l(\omega) \int dz \frac{e^{i\lambda\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}} \ . \tag{37}$$

This integral is given by

$$\int_1^\infty dx \frac{\sin ax}{\sqrt{x^2-1}} = \frac{\pi}{2} J_0(a) \ , \ \int_1^\infty dx \frac{\cos ax}{\sqrt{x^2-1}} = -\frac{\pi}{2} Y_0(a) \ , \tag{38}$$

where J_0 is the Bessel function of the first kind and zeroth order and Y_0 is the Bessel function of the second kind and zeroth order (Neumann functon); and $J_0 + iY_0 = H_0^{(1)}$ is the Bessel function of the third kind and zeroth order (Hankel function of the first kind). We get

$$A(\omega, \mathbf{r}) = -i\lambda p_l(\omega) \int dz \frac{e^{i\lambda\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}} = \pi\lambda p_l(\omega) H_0^{(1)}(\lambda r) \ . \tag{39}$$

We can see that $div\mathbf{A} = 0$ and $\Phi = 0$. The limiting behaviour of the Hankel function is

$$H_0^{(1)}(z) \sim \begin{cases} \sqrt{\frac{2}{\pi z}} e^{i(z-\pi/4)} \ , \ z \rightarrow \infty \\ \frac{2i}{\pi} \ln z \ , \ z \rightarrow 0 \ . \end{cases} \tag{40}$$

It is easy to see that we have a tangential magnetic field ($H_r = H_z = 0$)

$$H_\theta = -\frac{\partial A}{\partial r} \sim \begin{cases} -i\pi\lambda^2 p_l(\omega) \sqrt{\frac{2}{\pi\lambda r}} e^{i(\lambda r - \pi/4)} \ , \ \lambda r \rightarrow \infty \ , \\ -2i\lambda p_l(\omega)\frac{1}{r} \ , \ \lambda r \rightarrow 0 \end{cases} \tag{41}$$

and a z-component of the electric field ($E_r = 0$, $E_\theta = 0$)

$$E_z = i\lambda A \sim \begin{cases} i\pi\lambda^2 p_l(\omega) \sqrt{\frac{2}{\pi\lambda r}} e^{i(\lambda r - \pi/4)} \ , \ \lambda r \rightarrow \infty \ , \\ -2\lambda^2 p_l(\omega) \ln \lambda r \ , \ \lambda r \rightarrow 0 \ . \end{cases} \tag{42}$$

The average Poynting vector is given by

$$\overline{S}_r = \frac{1}{4\pi c^2 rT} \int d\omega \omega^3 \left| p_l(\omega) \right|^2 \ ; \tag{43}$$

multiplied by $2\pi r$ it gives the energy radiated per unit time and unit length.

A similar treatment can be done for a semi-infinite linear antenna (extending from $z = -\infty$ to $z = 0$), or a finite one.

We consider now a panel of linear antennas, extending indefinitely in the y, z-plane, oriented along the z-direction and of thickness a (along the x-direction), described by the displacement field

$$\mathbf{u}(t, x) = a(0, 0, u(t))\delta(x) \ . \tag{44}$$

Again $\rho = 0$ and $div\mathbf{j} = 0$, where the current density is

$$\mathbf{j}(t, x) = (0, 0, nqa\dot{u}(t))\delta(x) = (0, 0, \dot{p}_s(t))\delta(x) \ , \tag{45}$$

where $p_s(t) = nqau(t)$ is the surface density of dipoles. The vector potential is given by

$$A(\omega, x) = A_z(\omega, x) = -i\lambda p_s(\omega) \int d\mathbf{r} \frac{e^{i\lambda\sqrt{r^2+x^2}}}{\sqrt{r^2+x^2}} = 2\pi p_s(\omega) e^{i\lambda|x|} \ , \tag{46}$$

or

$$A(t,x) = 2\pi p_s(t - |x|/c) \tag{47}$$

(and $\Phi = 0$). We can see that the field consists of a superposition of plane waves,

$$E_z = -\frac{2\pi}{c}\dot{p}_s(t - |x|/c) \ , \ H_y = \frac{2\pi}{c}\dot{p}_s(t - |x|/c)sgn(x) \tag{48}$$

(all the other components are zero). The Poynting vector is given by

$$S_x = \frac{\pi}{c}\dot{p}_s^2(t - |x|/c)sgn(x) \ . \tag{49}$$

An interesting situation occurs for a semi-infinite space (half-space) of dipoles oriented along the z-direction, described by

$$\mathbf{u}(t,z) = (0,\, 0,\, u(t))\theta(-z) \ , \tag{50}$$

and

$$\rho = p(t)\delta(z) \ , \ \mathbf{j}(t) = (0,\, 0,\, \dot{p}(t))\theta(-z) \ , \tag{51}$$

where $p(t)$ is the density of dipoles (polarization). It is easy to get the potentials

$$\Phi(\omega, z) = -\frac{2\pi p(\omega)}{i\lambda}e^{i\lambda|z|} \tag{52}$$

and

$$A(\omega,z) = A_z(\omega,z) = -\frac{2\pi p(\omega)}{i\lambda}e^{i\lambda|z|}sgn(z) - \frac{4\pi p(\omega)}{i\lambda}\theta(-z) \ . \tag{53}$$

The magnetic field is vanishing and the electric field has only the z-component $E_z(\omega,z) = -4\pi p(\omega)\theta(-z)$ (vanishing outside).

Circular antennas. Consider a circular antenna of radius b and thickness a placed in the x, y-plane. The displacement field is given by

$$\mathbf{u}(t,\mathbf{R}) = (-u(t)\sin\varphi,\, u(t)\cos\varphi,\, 0)a^2\delta(r-b)\delta(z) \tag{54}$$

with usual notations in cylindrical coordinates. It is easy to check that $div\mathbf{u} = 0$, so that $\rho = 0$ (and $div\mathbf{j} = 0$). The current density can be writen as

$$\mathbf{j}(t,\mathbf{R}) = \dot{p}_l(t)(-\sin\varphi,\, \cos\varphi,\, 0)\delta(r-b)\delta(z) \ , \tag{55}$$

where $p_l(t) = nqa^2u(t)$ is the linear density of dipoles. The vector potential can be written as

$$\begin{gathered}\mathbf{A}(\omega,R) = 4\pi i\lambda b p_l(\omega)(-\sin\varphi,\, \cos\varphi,\, 0)\cdot \\ \cdot\int_0^\pi d\varphi' \frac{\cos\varphi'}{(R^2+b^2-2rb\cos\varphi')^{1/2}}e^{i\lambda(R^2+b^2-2rb\cos\varphi')^{1/2}} \ .\end{gathered} \tag{56}$$

We note that $2rb/(R^2+b^2) < 1$; in addition, we limit ourselves to $\lambda b \ll 1$. Then, we may expand the integrand and retain the first non-vanishing contribution. We get

$$\mathbf{A}(\omega,\mathbf{R}) = \begin{cases} 2\pi^2\lambda^2b^2p_l(\omega)\frac{e^{i\lambda R}}{R^2}(-y,\, x,\, 0) \ , \ \lambda R \gg 1 \ , \\ 2\pi^2 i\lambda b^2 p_l(\omega)\frac{1}{R'^3}(-y,\, x,\, 0) \ , \ \lambda R \ll 1 \ , \end{cases} \tag{57}$$

where $R' = (R^2+b^2)^{1/2}$ (and $\Phi = 0$). The electric field is given by

$$\mathbf{E}(\omega,\mathbf{R})=\begin{cases} 2\pi^2 i\lambda^3 b^2 p_l(\omega)\frac{e^{i\lambda R}}{R^2}(-y,\,x,\,0)\ ,\ \lambda R\gg 1\ , \\ -2\pi^2\lambda^2 b^2 p_l(\omega)\frac{1}{R'^3}(-y,\,x,\,0)\ ,\ \lambda R\ll 1 \end{cases} \tag{58}$$

and the magnetic field reads

$$\mathbf{H}(\omega,\mathbf{R})=\begin{cases} 2\pi^2 i\lambda^3 b^2 p_l(\omega)\frac{e^{i\lambda R}}{R^2}(-\frac{xz}{R},\,-\frac{yz}{R},\,\frac{r^2}{R})\ ,\ \lambda R\gg 1\ , \\ 2\pi^2 i\lambda b^2 p_l(\omega)\frac{1}{R'^5}(3xz,\,3yz,\,2R'^2-3r^2)\ ,\ \lambda R\ll 1\ . \end{cases} \tag{59}$$

We can see that $\mathbf{E}$ has only the tangential component, while this component is vanishing for the magnetic field. In addition $\mathbf{E}$ and $\mathbf{H}$ are orthogonal to each other; both are orthogonal to $\mathbf{R}$ in the wave zone; in the near-field zone $\mathbf{E}$ is orthogonal to $\mathbf{R}'$ (and much weaker than the magnetic field).

The Poynting vector (averaged over time) is given by

$$\overline{\mathbf{S}}=\frac{\pi^3 b^4}{c^5 R^2}\sin^2\theta\frac{1}{2\pi T}\int d\omega\omega^6\left|p_l(\omega)\right|^2\frac{\mathbf{R}}{R}\ . \tag{60}$$

We take now an infinite, uniform distribution of circular antenna (a solenoid), with the current density

$$\mathbf{j}(t,\mathbf{r})=\dot{p}_s(t)(-\sin\varphi,\,\cos\varphi,\,0)\delta(r-b)\ \ , \tag{61}$$

where $p_s=nqau(t)$ is the surface density of dipoles. Equation (56) contains now an integration over z, which, making use of equation (39), leads to a Hankel function. We get

$$\mathbf{A}(\omega,\mathbf{r})=-4\pi^2\lambda b p_s(\omega)(-\sin\varphi,\,\cos\varphi,\,0)\int_0^{\pi}d\varphi'\cos\varphi' H_0^{(1)}(\lambda\sqrt{r^2+b^2-2rb\cos\varphi'})\ . \tag{62}$$

Making use of the limiting behaviour of the Hankel function we can perform the integration in equation (). The vector potential is approximately

$$\mathbf{A}(\omega,\mathbf{r})=\begin{cases} 2\pi^3 i\lambda^2 b^2 p_s(\omega)\sqrt{\frac{2}{\pi\lambda r}}e^{i\lambda(r-\pi/4)}(-\frac{y}{r},\,\frac{x}{r},\,0)\ ,\ \lambda r\gg 1\ , \\ 4\pi^2 i\lambda b^2 p_s(\omega)(-\frac{y}{r'^2},\,\frac{x}{r'^2},\,0)\ ,\ \lambda r\ll 1\ \ , \end{cases} \tag{63}$$

where $r'=(r^2+b^2)^{1/2}$. The fields are given by

$$\mathbf{E}(\omega,\mathbf{r})=\begin{cases} -2\pi^3\lambda^3 b^2 p_s(\omega)\sqrt{\frac{2}{\pi\lambda r}}e^{i\lambda(r-\pi/4)}(-\frac{y}{r},\,\frac{x}{r},\,0)\ ,\ \lambda r\gg 1\ , \\ -4\pi^2\lambda^2 b^2 p_s(\omega)(-\frac{y}{r'^2},\,\frac{x}{r'^2},\,0)\ ,\ \lambda r\ll 1\ . \end{cases} \tag{64}$$

and

$$\mathbf{H}(\omega,\mathbf{r})=\begin{cases} -2\pi^3\lambda^3 b^2 p_s(\omega)\sqrt{\frac{2}{\pi\lambda r}}e^{i\lambda(r-\pi/4)}(0,\,0,\,1)\ ,\ \lambda r\gg 1\ , \\ 8\pi^2 i\lambda b^2 p_s(\omega)(0,\,0,\,\frac{b^2}{r'^4})\ ,\ \lambda r\ll 1\ . \end{cases} \tag{65}$$

We can see that the electric field has only the tangential component, while the magnetic field has only the z-component (and the electric field is much weaker than the magnetic field in the near-field zone). The Poynting vector is given by

$$\overline{\mathbf{S}}=\frac{2\pi^4 b^4}{c^4 r}\cdot\frac{1}{2\pi T}\int d\omega\omega^5\left|p_s(\omega)\right|^2\frac{\mathbf{r}}{r}\ . \tag{66}$$

We consider now an infinite tube of radius b with a surface density of dipoles p_s oriented along the z-axis of the tube. The current density is given by

$$\mathbf{j}(t,\mathbf{r})=(0,\,0,\,\dot{p}_s(t))\delta(r-b) \tag{67}$$

(and the charge density is zero). We find easily the vector potential

$$A(\omega, \mathbf{r}) = A_z(t, \mathbf{r}) = \pi\lambda b p_s(\omega) \int d\varphi H_0^{(1)}(\lambda\sqrt{r^2 + b^2 - 2rb\cos\varphi}) \simeq$$

$$\simeq \begin{cases} 2\pi^2\lambda b p_s(\omega)\sqrt{\frac{2}{\pi\lambda r}}e^{i(\lambda r - \pi/4)} \ , \ \lambda r \gg 1 \ , \\ 4\pi i\lambda b p_s(\omega)\ln(\lambda\sqrt{r^2 + b^2}) \ , \ \lambda r \ll 1 \ . \end{cases} \tag{68}$$

We get a z-component of the electric field

$$E_z(\omega, \mathbf{r}) \simeq \begin{cases} 2\pi^2 i\lambda^2 b p_s(\omega)\sqrt{\frac{2}{\pi\lambda r}}e^{i(\lambda r - \pi/4)} \ , \ \lambda r \gg 1 \ , \\ -4\pi\lambda^2 b p_s(\omega)\ln(\lambda\sqrt{r^2 + b^2}) \ , \ \lambda r \ll 1 \end{cases} \tag{69}$$

and a tangential component of the magnetic field

$$\mathbf{H}(\omega, \mathbf{r}) \simeq \begin{cases} 2\pi^2 i\lambda^2 b p_s(\omega)\sqrt{\frac{2}{\pi\lambda r}}e^{i(\lambda r - \pi/4)}(\frac{y}{r}, -\frac{x}{r}, 0) \ , \ \lambda r \gg 1 \ , \\ 4\pi i\lambda b p_s(\omega)(\frac{y}{r'^2}, -\frac{x}{r'^2}, 0) \ , \ \lambda r \ll 1 \ , \end{cases} \tag{70}$$

where $r' = (r^2 + b^2)^{1/2}$. The Poynting vector is

$$\overline{\mathbf{S}} = \frac{2\pi^3 b^2}{c^2 r} \cdot \frac{1}{2\pi T}\int d\omega\omega^3 \left|p_s(\omega)\right|^2 \frac{\mathbf{r}}{r} \ . \tag{71}$$
