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Propagation of electromagnetic pulses through the surface of dispersive bodies

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Abstract

The motion of an electromagnetic pulse (signal) through the surface of a semi-infinite (half-space) polarizable body is investigated. The incident pulse of electromagnetic radiation propagating in vacuum is assumed to be of finite duration and finite spatial extension. As regards its extension along the transverse directions, two cases are considered. First, we assume a large (infinite) extension (in comparison with the wavelength), as for a plane wave (beam, ray); second, a very narrow pulse is assumed (zero thickness, close to the difraction limit). In its motion the pulse encounters the plane surface of a semi-infinite polarizable body (a half-space) and penetrates into the body. The body reacts through its polarization degrees of freedom, wich obey the well-known Drude-Lorentz (plasma) equation of motion. It is shown that the beam obeys the well-known refraction law (Fresnel equations), with a specific discussion, which is provided. For the narrow pulse, both the normal and oblique incidence are analyzed. It is shown that far away from the incidence direction (large transverse distance r) the motion is governed by the polaritonic eigenmodes, which yields a pulse, approximately of the same shape as the original one, propagating with the group velocity and with an amplitude which decreases as $1/r^2$. The group velocity is always smaller than the speed of light in vacuum c. In the vicinity of the propagation direction (small distance r), the original pulse is almost entirely preserved, including its propagation velocity c, with a distorted amplitude, which depends on the transverse direction. This picture is in fact the diffracton limit of the narow pulse. The transmitted coefficient is computed for normal incidence. The reflected pulse is also computed, as well as the refracted pulse for oblique incidence. While the reflection law is preserved (reflection angle is equal to the incidence angle), the refraction law is different from Snell's law of refraction of a plane wave, in the sense that the highly localized (narrow) pulse along the transverse direction preserves its propagation direction on entering into the body.

Introduction. In 1914, in two classic papers, Sommefeld and Brillouin analyzed the propagation of an electromagnetic pulse (signal) in dispersive matter, in the context of a phase velocity which, in some cases, trespasses the speed of light[1, 2] (see also Refs. [3, 4]). Making use of the saddle point method,[5] the group velocity has been introduced on this occasion, and non-locality of the electromagnetic waves, sometimes associated with velocities higher than the speed of light, has been highlighted for precursors (forerunners). In addition, it was shown that the front of the signal propagates with the speed of light in vacuum. The treatment has been refined in modern times, including experimental measurements of the propagation velocity.[6]-[9]

The treatment is usually simplified, in comparison with the experimental situation which is more complex. For instance, the dependence on the transverse coordinates with respect to the propagation direction of the pulse is very helpful in getting a more complete picture. Second, the presence of the surface through which the pulse penetrates into the body brings new realistic features. Not in the least, the motion of the polarization degrees of freedom of the body exhibits a more complex dynamics (for instance inducing a longitudinal response) than the usual account based on a model dielectric function. Such more realistic features are incorporated here, by making use of a method previously introduced for including the motion of the polarization.[10] Apart form their fundamental interest, the results obtained here may also be of interest for the current experimental investigations of electrons accelerated by focalizing laser beams in rarefied plasmas (see, for instance, Refs. [11, 12].

Uusally, the classical treatment rests upon assuming a radiation beam (a ray), of finite temporal duration (and finite spatial extension), produced at some point in the body and followed in its propagation through the body. The finite duration entails a superposition of frequencies and, since the body is dispersive, a corresponding dependence of the propagation wavevector on frequency. Consequently, the propagation lends itself naturally to be analyzed in terms of the group velocity, with all its limitations and particularities. We introduce here two different features. First, we asssume that the pulse of finite duration and spatial extension penetrates into the body through a plane surface, either at normal or oblique incidence. The presence of the surface for a semi-infinite (half-space) polarizable body allows the description of the usual reflection and refraction laws for a radiation beam (ray),[10] for a pulse of finite duration and extension in the present context. Next, with the advent of highly-localized laser beams focalized in plasma, it is worth investigating a pulse of a vanishing thickness, *i.e.* a δ-pulse along the transverse directions (narrrow pulse). This is the second distinct feature we introduce in the present paper.

We find that such a zero-thickness electromagnetic pulse enters the body from vacuum almost in its entirety, *i.e.* it preserves its shape to a large extent; it preserves also the propagation direction and velocity, while suffering a distortion in amplitude. At the same time, the pulse produces an electromagnetic disturbance both in the body and outside it (a reflected field), which is extended spatially along the transverse directions, is vanishing at infinity as the inverse square of the transverse distance, has a finite duration and spatial extension along the propagation direction, is governed by the polaritonic eigenmodes and propagates with the group velocity. This picture corresponds in fact to the diffraction limit of the narrow pulse. The calculations are performed within the well-known Drude-Lorentz (plasma) model of polarizable (non-magnetic) matter.

Polarization eigenmodes. A usual model of polarizable matter assumes the existence of (point) mobile charges q, with mass m and concentration n (*e.g.*, electrons), subjected to a displacement field $\mathbf{u}(t,\mathbf{R})$, which is a function of the time t and position $\mathbf{R}$. The displacement field produces a slight imbalance $\delta n = -ndiv\mathbf{u}$ in the particles density, a charge density $\rho = -nqdiv\mathbf{u}$ and a current density $\mathbf{j} = nq\dot{\mathbf{u}}$. These charge and current densities give rise to an electric field $\mathbf{E}$ and a magnetic field $\mathbf{H}$ wich, for a non-magnetic matter, obey the Maxwell equations

$$\begin{gathered} div\mathbf{E} = 4\pi\rho = -4\pi nqdiv\mathbf{u} \ , \ div\mathbf{H} = 0 \ , \\ curl\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} \ , \ curl\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}nq\dot{\mathbf{u}} \end{gathered} \tag{1}$$

(where c denotes the speed of light). We can see that $\mathbf{P} = nq\mathbf{u}$ is the polarization (and the fields $\mathbf{E}$ and $\mathbf{H}$ are internal, or polarization fields of the body). Only two equations (1) are independent (those involving time derivatives), while there are three unknowns ($\mathbf{E}$, $\mathbf{H}$ and $\mathbf{u}$) in equations (1). The third (missing) equation is provided by the equation of motion

$$m\ddot{\mathbf{u}} = q(\mathbf{E} + \mathbf{E}_0) - m\omega_c^2\mathbf{u} - m\gamma\dot{\mathbf{u}} \tag{2}$$

for the displacement field $\mathbf{u}$, which is Newton's equation for the charges motion. In equation (2) $\mathbf{E}_0$ is an external electric field, ω_c is a characteristic frequency (for "bound" charges) and

γ is a damping parameter (much smaller than any relevant frequency). The contribution of the magnetic field to the law of motion (Lorentz force) is left aside, in accordance with the assumption of non-magnetic matter, in view of the fact that the particle velocities are much smaller than the speed of light and according to the hypothesis of small magnitude u of the displacement field in comparison with the relevant wavelengths. The Fourier transform of equatiom (2) with respect to time leads to the well-known electric susceptibility $\chi(\omega)$ and dielectric function

$$\varepsilon(\omega) = 1 + 4\pi\chi(\omega) = \frac{\omega^2 - \omega_c^2 - \omega_p^2}{\omega^2 - \omega_c^2 + i\omega\gamma} \ , \tag{3}$$

where $\omega_p = \sqrt{4\pi nq^2/m}$ is called the plasma frequency. This is the well-known Drude-Lorentz (plasma) model of polarizable matter (dispersive bodies).[13]-[15] As it is well known, for conductors $\omega_c = 0$, for dielectrics $\omega_c \neq 0$ (and, usually, $\omega_p \ll \omega_c$, *i.e.* the band $\omega_c < \omega < \sqrt{\omega_c^2 + \omega_p^2}$ of anomalous dispersion is narrow); ω_c is called the frequncy of transverse modes ($\omega_T = \omega_c$), while $\omega_L = \sqrt{\omega_c^2 + \omega_p^2}$ is called the frequency of the longitudinal modes.[16, 17] According to its definition, the conductivity is $\sigma(\omega) = -i\omega\chi(\omega)$.

For an infinite body it is convenient to use the spatial Fourier transforms, with the wavevector $\mathbf{K}$; it is also convenient to use the longitudinal projection of the vectors along the wavevector $\mathbf{K}$, denoted by subscript 1, and the transverse projection of the vectors, *i.e.* their projection perpendicular to the wavevector $\mathbf{K}$, denoted by subscript 2. Maxwell equations (1) supplemented by equation (2) can then be solved straighforwardly, and we get the polarizabilities

$$\begin{gathered}\alpha_1 = P_1/E_{01} = -\frac{\omega_p^2}{4\pi}\frac{1}{\omega^2-\omega_c^2-\omega_p^2+i\omega\gamma} \ , \\ \alpha_2 = P_2/E_{02} = -\frac{\omega_p^2}{4\pi}\frac{\lambda^2-K^2}{(\omega^2-\omega_c^2)(\lambda^2-K^2)-\omega_p^2\lambda^2+i\omega(\lambda^2-K^2)\gamma} \ ,\end{gathered} \tag{4}$$

where $\lambda = \omega/c$. The polarizations $P_{1,2} = nqu_{1,2}$ are related to the displacements $u_{1,2}$ and the latter, by equation (2), give the total electric field $E_{t1,2} = E_{1,2} + E_{01,2} = -(m/q)(\omega^2 - \omega_c^2 + i\omega\gamma)u_{1,2}$. The corresponding magnetic field is obtained from the Faraday equation $curl\mathbf{E} = -(1/c)\partial\mathbf{H}/\partial t$.

The first thing we notice in equations (4) is the fact that a free transverse wave cannot be propagated in matter, since, for $\lambda = K$ ($\omega = cK$), the total transverse electric field is vanishing (second eqaution (4)). This is the well-known Ewald-Oseen extinction theorem.[18]-[20] It is worth emphasizing this peculiarity of the propagation of the electromagnetic field in matter (known for a long time), since, although it implies a vanishing (transverse) displacement field $\mathbf{u}$ and a vanishing (transverse) total field, there still exists a polarization (internal) field which compensates exactly the external field. In the presence of a surface (as for a half-space, for instance) the polarization field gives rise to the refracted ray.[10]

Next, we notice in equations (4) the singularities arising from the vanishing of the denominators; they define the polarization eigenmodes of matter. From the first equation (4) we get the longitudinal mode

$$\Omega_1 = \omega_L = \sqrt{\omega_c^2 + \omega_p^2} \tag{5}$$

(which is non-dispersive); it is usually called the plasmon mode. From the second equation (4) we get, in general, two transverse eigenmodes, $\Omega_{2,3}(K)$, which are dispersive; they are given by the roots of the well-known equation $\varepsilon(\omega)\lambda^2 = K^2$. In the long-wavelength limit they go like

$$\Omega_2(K) \simeq \sqrt{\omega_L^2 + c^2K^2} \ , \ \Omega_3(K) = vK \ , \ v = c\omega_c/\omega_L \ , \tag{6}$$

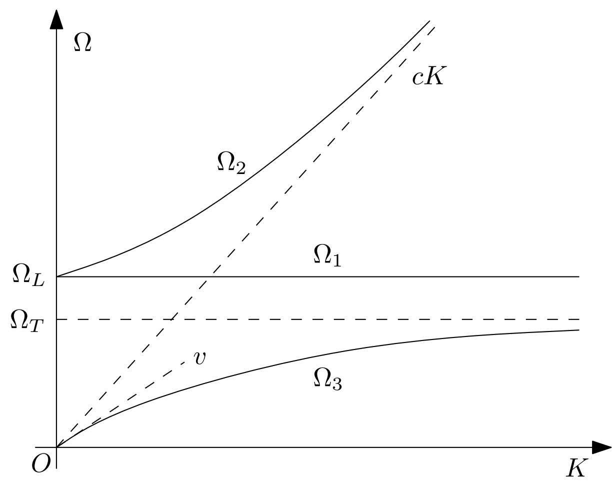

Figure 1: The polarization longitudinal eigenmode Ω_1 and the two transverse eigenmodes $\Omega_{2,3}$. We note the velocity v of the "renormalized" transverse waves (cK is the frequency of the free transverse waves).

while in the short-wavelength limit they behave as $\Omega_2(K) \simeq cK$ and $\Omega_3(K) \simeq \omega_c = \omega_T$. They are usually called the polaritonic modes. All these eigenmodes are shown in Fig. 1. We note the "renormalization" of the speed of light $c \rightarrow v$ for the second polaritonic mode $\Omega_3(K)$. It is worth emphasizing that the polarization eigenmodes are different in different conditions (for instance in the presence of a surface, or in the presence of an external uniform magnetic field, etc). The propagation of electromagnetic waves in matter is governed by the polaritonic frequencies. It was the group velocity $\mathbf{v}_{g2,3} = \partial\Omega_{2,3}/\partial\mathbf{K}$ which was identified in Refs. [1]-[4] for the propagation of light in dispersive media. Noteworthy, it is always smaller than the speed of ligh in vacuum c.

External pulse. We consider an electric field

$$E_0(t,\mathbf{R}) = E_{0x}(t,\mathbf{R}) = E_0 \cos\omega_0(t - z/c)\theta(ct - z)\theta(z - ct + d) \cdot d_t^2\delta(\mathbf{r}) \ , \tag{7}$$

where ω_0 is the main frequency, d is the length of the pulse along the propagation z-axis and d_t is the transverse dimension of the pulse. This is a very narrow pulse. We use the notations $\mathbf{R} = (\mathbf{r}, z)$ (and $\mathbf{K} = (\mathbf{k}, \kappa)$); $\theta(z) = 1$ for $z > 0$ and $\theta(z) = 0$ for $z < 0$ is the step function and $\delta(r)$ is the Dirac δ-function. The associated magnetic field is $H_{0y} = E_{0x}$. The pulse carries the density $E_0^2/8\pi$ of electromagnetic energy along the z-xis with velocity c. Equation (7) can be considered as giving an adequate representation of an electromagnetic pulse (signal) of a vanishing thickness (δ-function along the transverse directions). Due to its limitation in time (and space) the pulse given by equation (7) contains many other frequencies beside the main frequency ω_0, as well as many other wavelengths beside the main wavelength c/ω_0. Inded, the Fourier transform

$$E_0(t,\mathbf{r},z) = \frac{1}{(2\pi)^3}\int d\omega \int d\mathbf{k} E_0(\omega,\mathbf{k};z)e^{-i\omega t}e^{i\mathbf{kr}} \tag{8}$$

gives

$$E_0(\omega,\mathbf{k};z) = E_0(\omega)d_t^2 e^{i\omega z/c} \ , \tag{9}$$

where

$$E_0(\omega) = E_0\left[e^{i(\omega-\omega_0)d/2c}\frac{\sin(\omega-\omega_0)d/2c}{\omega-\omega_0} + (\omega_0 \rightarrow -\omega_0)\right] \ . \tag{10}$$

We can see that the pulse consists of a superposition of frequencies in a band of approximate half-width $\Delta\omega \simeq 2\pi c/d$ around the main frequencies $\pm\omega_0$; a further Fourier transform with respect to the coordinate z gives $\omega = c\kappa$, which implies a similar superposition of wavelengths; this is a well-known wavepacket. In the limit $d \to \infty$ it becomes a superposition of two monochromatic waves, $E_0(\omega) = \pi E_0[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$, while in the opposite limit $d \to 0$ we get a δ-pulse $E_0(\omega) = E_0 d/c$, *i.e.* $E_0(t, \mathbf{r}, z) = E_0 V \delta(z - ct)\delta(\mathbf{r})$, where $V = dd_t^2$ is the volume of the pulse.

In the subsequent discussion we describe the propagation of such an electromagnetic pulse in a polarizable half-space, *i.e.* a (homogeneous) semi-infinite body with a plane surface at $z = 0$. We consider first the normal incidence, which contains the most relevant features; the oblique incidence is treated thereafter, especially with the aim of investigating the law of pulse refraction and reflection.

A technical note is worth introducing here. The function in equation (10) is finite for $\omega = \omega_0$. However, we may need to integrate over ω the expression of $\sin(\omega - \omega_0)d/2c$ with exponentials, in which case we get functions of the type $(\omega \pm \omega_0)^{-1}$, which have singularities (poles) at $\omega = \pm\omega_0$. In this case, in order to have a "good causality", we should take the poles as being placed in the lower half-plane, corresponding to $\omega \to \pm\omega_0 - i0^+$, or $\omega \to \omega + i0^+$; indeed, the temporal factors $e^{-i\omega t}$ appearing in such integrations over ω have a good behaviour for the "past" contributions corresponding to $t < 0$. This point is emphasized in Ref. [1, 2] (see also Ref. [21]).

Half-space. The motion of the polarization in a half-space was given in Refs. [10, 22]. Here we include a few results from Ref. [22] which are necessary for describing the propagation of an electromagnetic pulse. The polarization for a half-space is taken as

$$\mathbf{P} = nq(\mathbf{u},\, u_z)\theta(z) \ , \tag{11}$$

giving rise to charge and current densities

$$\begin{gathered} \rho = -nq(div\mathbf{u} + \tfrac{\partial u_z}{\partial z})\theta(z) - nqu_z(z=0)\delta(z) \ , \\ \mathbf{j} = nq(\dot{\mathbf{u}},\, \dot{u}_z)\theta(z) \ . \end{gathered} \tag{12}$$

We use Fourier decompositions of the type given by equation (8) and may omit ocassionally the arguments $\mathbf{k}$, ω, writing simply $\mathbf{u}(z)$, or $\mathbf{u}$, for instance. We compute the electromagnetic potentials given by

$$\begin{gathered} \Phi(t, \mathbf{R}) = \int d\mathbf{R}' \tfrac{\rho(t - |\mathbf{R} - \mathbf{R}'|/c, \mathbf{R}')}{|\mathbf{R} - \mathbf{R}'|} \ , \\ \mathbf{A}(t, \mathbf{R}) = \tfrac{1}{c} \int d\mathbf{R}' \tfrac{\mathbf{j}(t - |\mathbf{R} - \mathbf{R}'|/c, \mathbf{R}')}{|\mathbf{R} - \mathbf{R}'|} \end{gathered} \tag{13}$$

by using the well-known decomposition[23]

$$\frac{e^{i\lambda|\mathbf{R} - \mathbf{R}'|}}{|\mathbf{R} - \mathbf{R}'|} = \frac{i}{2\pi} \int d\mathbf{k} \frac{1}{\kappa} e^{i\mathbf{k}(\mathbf{r} - \mathbf{r}')} e^{i\kappa|z - z'|} \ , \tag{14}$$

for the "retarded" Coulomb potential $e^{i\frac{\omega}{c}|\mathbf{R} - \mathbf{R}'|}/\,|\mathbf{R} - \mathbf{R}'|$, where $\lambda = \omega/c$ and $\kappa = \sqrt{\lambda^2 - k^2}$. It is more convenient to compute first the vector potential $\mathbf{A}$ and then derive the scalar potential Φ from the gauge equation $div\mathbf{A} - i\lambda\Phi = 0$. The calculations are straightforward and we get the Fourier tranforms of the potentials

$$\begin{gathered} \Phi(\omega, \mathbf{k}; z) == \tfrac{2\pi}{\kappa} \int_0^\infty dz' \mathbf{k}\mathbf{u} e^{i\kappa|z - z'|} - \tfrac{2\pi i}{\kappa} \tfrac{\partial}{\partial z} \int_0^\infty dz' u_z e^{i\kappa|z - z'|} \ , \\ \mathbf{A}(\omega, \mathbf{k}; z) = \tfrac{2\pi\lambda}{\kappa} \int_0^\infty dz' (\mathbf{u},\, u_z) e^{i\kappa|z - z'|} \ , \end{gathered} \tag{15}$$

where we leave aside the factor nq; it is restored in the final formulae. In order to compute the electric field ($\mathbf{E}=i\lambda\mathbf{A}-grad\Phi$) it is convenient to use the vectors $\mathbf{k}$ and $\mathbf{k}_\perp=e_z\times\mathbf{k}$ as reference axes, where e_z is the unit vector along the z-direction; for instance, we write

$$\mathbf{u}=u_1\frac{\mathbf{k}}{k}+u_2\frac{\mathbf{k}_\perp}{k} \tag{16}$$

as above, and a similar representation for the electric field parallel with the surface of the half-space. We note that in order to satisfy the condition $\mathbf{u}^*(-\omega,-\mathbf{k};z)=\mathbf{u}(\omega,\mathbf{k};z)$ the coordinates $u_{1,2}$ should include a factor $\pm i$; we choose $-i$. We get the electric field

$$\begin{gathered}E_1=2\pi i\kappa\int_0^\infty dz'u_1e^{i\kappa|z-z'|}-\tfrac{2\pi k}{\kappa}\tfrac{\partial}{\partial z}\int_0^\infty dz'u_ze^{i\kappa|z-z'|}\ ,\\ E_2=\tfrac{2\pi i\lambda^2}{\kappa}\int_0^\infty dz'u_2e^{i\kappa|z-z'|}\ ,\\ E_z=-\tfrac{2\pi k}{\kappa}\tfrac{\partial}{\partial z}\int_0^\infty dz'u_1e^{i\kappa|z-z'|}+\tfrac{2\pi ik^2}{\kappa}\int_0^\infty dz'u_ze^{i\kappa|z-z'|}-4\pi u_z\theta(z)\ .\end{gathered} \tag{17}$$

Making use of equations (17), we can check easily the equalities

$$ikE_1+\frac{\partial E_z}{\partial z}=-4\pi\left(iku_1+\frac{\partial u_z}{\partial z}\right)\theta(z)-4\pi u_z(z=0)\delta(z)\ , \tag{18}$$

which is an expression of Gauss's law, and

$$k\frac{\partial E_1}{\partial z}+i\kappa^2E_z=-4\pi i\lambda^2u_z\theta(z)\ , \tag{19}$$

which reflects Faraday's and Maxwell-Ampere's equations. From equation (18), we can check the transversality condition $div\mathbf{E}=0$ for the electric field outside the half-space ($z<0$). The magnetic field can be obtained either from $\mathbf{H}=curl\mathbf{A}$ or from $curl\mathbf{E}=(-1/c)\partial\mathbf{H}/\partial t$.

Fresnel equations. We use now the equation of motion (2) (with $\gamma=0$) for E_2 given by equation (17) and for the combinations $iku_1+\partial u_z/\partial z$ and $k\partial u_1/\partial z+i\kappa^2u_z$ in the region $z>0$. In general, we get equations which contains contributions from the external field $\mathbf{E}_0$. Assuming that $\mathbf{E}_0$ is a radiation field we have $div\mathbf{E}_0=0$ and $k\partial E_{01}/\partial z+i\kappa^2E_{0z}=0$ (for a plane wave) and

$$iku_1+\frac{\partial u_z}{\partial z}=0\ ,\ k\frac{\partial u_1}{\partial z}+i\kappa'^2u_z=0\ , \tag{20}$$

or

$$\tfrac{\partial^2\mathbf{u}}{\partial z^2}+\kappa'^2\mathbf{u}=0\ , \tag{21}$$

where

$$\kappa'^2=\kappa^2-\frac{\lambda^2\omega_p^2}{\omega^2-\omega_c^2}\ . \tag{22}$$

The components $u_{1,2}$ of the displacement field are given by $u_{1,2}=A_{1,2}e^{i\kappa'z}$, where $A_{1,2}$ are constants, while $u_z=-(k/\kappa')A_1e^{i\kappa'z}$ (we restrict ourselves to outgoing waves, $\kappa'>0$). The total electric field inside the half-space is given by the equation of motion (2):

$$\mathbf{E}_t=-\frac{m}{q}(\omega^2-\omega_c^2)\mathbf{u} \tag{23}$$

for $z>0$. We can see that the field propagates in the half-space with a modified wavevector κ', according to the Ewald-Oseen extinction theorem. The modified wavevector κ' given by equation (22) can also be written as

$$\kappa'^2=\varepsilon\lambda^2-k^2=\varepsilon\frac{\omega^2}{c^2}-k^2\ , \tag{24}$$

where ε is the dielectric function (as given by equation (3)). We can check the well-known polaritonic dispersion relation $\varepsilon\omega^2 = c^2K'^2$, where $\mathbf{K}' = (\mathbf{k},\, \kappa')$ is the wavevector.

The amplitudes $A_{1,2}$ can be derived from the original equation (2) and the field equations (17) (for $z > 0$). We get

$$\begin{aligned} \tfrac{1}{2}A_1\omega_p^2\tfrac{\kappa\kappa'+k^2}{\kappa'(\kappa'-\kappa)}e^{i\kappa z} &= \tfrac{q}{m}E_{01}\ , \\ \tfrac{1}{2}A_2\omega_p^2\tfrac{\lambda^2}{\kappa(\kappa'-\kappa)}e^{i\kappa z} &= \tfrac{q}{m}E_{02}\ . \end{aligned} \tag{25}$$

The (polarization) electric field, both inside and outside the half-space, can be computed from equations (17). We get

$$\begin{aligned} E_1 &= -4\pi nqA_1\tfrac{\omega^2-\omega_c^2}{\omega_p^2}e^{i\kappa' z} - 2\pi nqA_1\tfrac{\kappa\kappa'+k^2}{\kappa'(\kappa'-\kappa)}e^{i\kappa z}\ ,\ z>0\ , \\ E_2 &= -4\pi nqA_2\tfrac{\omega^2-\omega_c^2}{\omega_p^2}e^{i\kappa' z} - 2\pi nqA_2\tfrac{\lambda^2}{\kappa(\kappa'-\kappa)}e^{i\kappa z}\ ,\ z>0\ , \\ E_z &= 4\pi nqA_1\tfrac{k(\omega^2-\omega_c^2)}{\kappa'\omega_p^2}e^{i\kappa' z} + 2\pi nqA_1\tfrac{k(\kappa\kappa'+k^2)}{\kappa\kappa'(\kappa'-\kappa)}e^{i\kappa z}\ ,\ z>0\ . \end{aligned} \tag{26}$$

for $z > 0$. It is worth noting that the polarization electric field, as given by equations (26), includes both the external field $\sim e^{i\kappa z}$ (with opposite sign) and the displacement field $\mathbf{u} \sim e^{i\kappa' z}$. This can be checked easily by using equations (25) and (26). The (polarization) electric field outside the half-space (in the region $z < 0$) is given by

$$\begin{aligned} E_1 &= -2\pi nqA_1\tfrac{\kappa\kappa'-k^2}{\kappa'(\kappa'+\kappa)}e^{-i\kappa z}\ ,\ z<0\ , \\ E_2 &= -2\pi nqA_2\tfrac{\lambda^2}{\kappa(\kappa'+\kappa)}e^{-i\kappa z}\ ,\ z<0 \end{aligned} \tag{27}$$

and $E_z = (k/\kappa)E_1$ for $z < 0$. We can see that it is the field reflected by the half-space ($\kappa \to -\kappa$). Making use of equations (25) and (27) we get the total electric field $\mathbf{E}_t = \mathbf{E} + \mathbf{E}_0$ outside the half-space, as well as the associated magnetic field everywhere.

According to the results given above a half-space subjected to an external radiation field

$$\mathbf{E}_0(\omega;\mathbf{k},z) = \mathbf{E}_0 e^{i\kappa z}\ \ , \tag{28}$$

where $\mathbf{E}_0$ is the amplitude of the (transverse) electric plane wave, develops an electric field ($z > 0$)

$$\begin{aligned} E_{t1}(\omega;\mathbf{k},z) &= -\tfrac{\omega^2-\omega_c^2}{\omega_p^2}\cdot\tfrac{2\kappa'(\kappa'-\kappa)}{\kappa\kappa'+k^2}E_{01}e^{i\kappa' z}\ , \\ E_{t2}(\omega;\mathbf{k},z) &= -\tfrac{\omega^2-\omega_c^2}{\omega_p^2}\cdot\tfrac{2\kappa(\kappa'-\kappa)}{\lambda^2}E_{02}e^{i\kappa' z}\ , \\ E_{tz}(\omega;\mathbf{k},z) &= \tfrac{\omega^2-\omega_c^2}{\omega_p^2}\cdot\tfrac{2k(\kappa'-\kappa)}{\kappa\kappa'+k^2}E_{01}e^{i\kappa' z} \end{aligned} \tag{29}$$

(where $kE_{01} + \kappa E_{0z} = 0$) and the associated magnetic field (which can be obtained from $curl\mathbf{E}_t = -(1/c)\partial\mathbf{H}_t/\partial t$); we can see that it is a transverse plane wave with the wavevector component κ normal to the surface changed into κ'. This is the refracted field, which obeys the extinction theorem. We note that the external field goes like $e^{i\kappa z}$, while the displacement field and the polarization go like $e^{i\kappa' z}$, which makes it difficult to define a polarizability (it becomes local). Snell's refraction law reads $\sin r/\sin i = K/K' = 1/\sqrt{\varepsilon}$ (for ω satisfying $\kappa' = \sqrt{\varepsilon\lambda^2 - k^2}$, *i.e.*

$\varepsilon\lambda^2 = K'^2$). It is worth noting that the denominators in equations (29) do not vanish for those ω where the denominators of equations (4) for infinite matter do; *i.e.*, the polarization eigenmodes are different for a half-space than for infinite matter. We get an eigenmode in equations (29) for both κ and κ' purely imaginary, which means that both the incident and the refracted waves are localized at the surface and propagates only along the surface. We call this mode a surface plasmon-polariton mode. Its frequency is given by

$$\omega^2 = \frac{2(2\omega_c^2 + \omega_p^2)c^2k^2}{\omega_c^2 + \omega_p^2 + 2c^2k^2 + \sqrt{(\omega_c^2 + \omega_p^2 + 2c^2k^2)^2 - 4(2\omega_c^2 + \omega_p^2)c^2k^2}} \tag{30}$$

for $c^2k^2 > \omega_c^2$; it goes from ω_c ($ck = \omega_c$) to $\sqrt{\omega_c^2 + \omega_p^2/2}$ ($k \to \infty$).

The component denoted by 2 corresponds to the s- ("senkrecht") wave (*i.e.* the wave whose polarization vector is perpendicular to the incidence plane). Since $\kappa'^2 = \kappa^2 - \lambda^2\omega_p^2/(\omega^2 - \omega_c^2)$ we get from equations (29)

$$E_{t2}(\omega; \mathbf{k}, z) = \frac{2\kappa}{\kappa' + \kappa} E_{02} e^{i\kappa' z} \ ; \tag{31}$$

we can compute the corresponding magnetic field (from $curl\mathbf{E}_t = -(1/c)\partial\mathbf{H}_t/\partial t$) and get the Poynting vector; the transmission coefficient is the ratio of the z-components of the refracted (transmitted) to the incident Poynting vectors, *i.e.*

$$T_s = \frac{\kappa'}{\kappa} \left|E_{t2}/E_{02}\right|^2 = \left|\frac{4\kappa\kappa'}{(\kappa' + \kappa)^2}\right| = \left|\frac{4\sqrt{\varepsilon}\cos i \cos r}{(\cos i + \sqrt{\varepsilon}\cos r)^2}\right| \ , \tag{32}$$

where we use $\kappa = K\cos i$, $\kappa' = K'\cos r$ and $\sin r / \sin i = K/K' = 1/\sqrt{\varepsilon}$.

The component denoted by 1 and the z-component correspond to the p-wave (the polarization is parallel with the incidence plane). Making use of equations (29) we compute the magnetic field for the p-wave and the z-component of the Poynting vector; its ratio to the z-component of the Poynting vector of the incident wave gives the transmission coefficient

$$T_p = \left|\frac{4\kappa\kappa'\varepsilon\lambda^4}{(\kappa + \kappa')^2(\kappa\kappa' + k^2)^2}\right| = \left|\frac{4\kappa\kappa'\varepsilon}{(\varepsilon\kappa + \kappa')^2}\right| = \left|\frac{4\sqrt{\varepsilon}\cos i \cos r}{(\sqrt{\varepsilon}\cos i + \cos r)^2}\right| \ . \tag{33}$$

The reflection coefficients are given by $R_{s,p} = 1 - T_{s,p}$; they can be obtained directly from the reflected field ($z < 0$)

$$\begin{aligned} E_1(\omega; \mathbf{k}, z) &= -\tfrac{(\kappa' - \kappa)(\kappa\kappa' - k^2)}{(\kappa' + \kappa)(\kappa\kappa' + k^2)} E_{01} e^{-i\kappa z} \ , \\ E_2(\omega; \mathbf{k}, z) &= -\tfrac{\kappa' - \kappa}{\kappa' + \kappa} E_{02} e^{-i\kappa z} \ , \end{aligned} \tag{34}$$

$E_z = kE_1/\kappa$, given by equations (27); we get

$$R_s = \left|\tfrac{\cos i - \sqrt{\varepsilon}\cos r}{\cos i + \sqrt{\varepsilon}\cos r}\right|^2 \ , \ R_p = \left|\tfrac{\sqrt{\varepsilon}\cos i - \cos r}{\sqrt{\varepsilon}\cos i + \cos r}\right|^2 \ . \tag{35}$$

These are Fresnel's formulae ($\sim$ 1820). It is worth noting that the reflected p-wave is vanishing for $\kappa\kappa' - k^2 = 0$, which means $\tan^2 i = \varepsilon$ and $R_p = 0$; this gives the angle of total polarization (Brewster's angle), because it remains reflected only the s-wave (which is "totally polarized"). The total reflection is obtained for $r = \pi/2$ (*i.e.* $\sin i = \sqrt{\varepsilon}$), where the transmission coefficients are vanishing (transparency edge). The dependence of the reflection and transmission coefficients on the incidence angle and frequency exhibits interesting features, especially related to sharp shoulders.

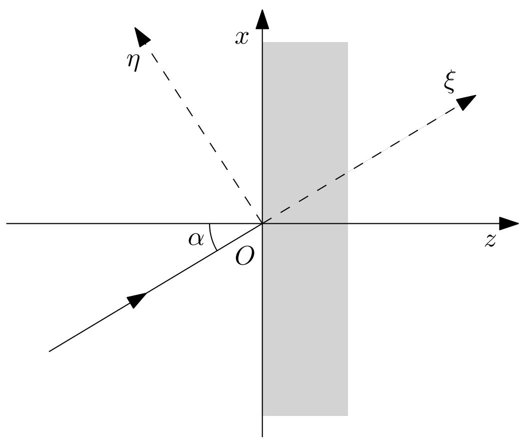

Figure 2: Pulse propagating along the ξ-direction against a half-space.

Pulsed plane wave (beam, ray).

We consider a ξ, η, y-frame associated with a half-space, as in Fig. 2, with the transformation of coordinates

$$x = \xi \sin\alpha + \eta\cos\alpha \ , \ z = \xi\cos\alpha - \eta\sin\alpha \ ,$$
$$\xi = x\sin\alpha + z\cos\alpha \ , \ \eta = x\cos\alpha - z\sin\alpha \tag{36}$$

(rotation by angle α about the y-axis). We consider also a radiation pulse with an electric field

$$E_0(t, \mathbf{R}) = E_{0\eta}(t, \mathbf{R}) = E_0\cos\omega_0(t - \xi/c)\theta(\xi - ct + d)\theta(ct - \xi) \ ; \tag{37}$$

it is a pulse of length d, with the main fequency ω_0, propagating in vacuum with phase velocity c, along the ξ-direction, with an incidence angle α. It is a p-wave; similarly, we can take an s-wave, with the electric field oriented along the y-axis. It is worth noting that the field does not depend on the transverse cooordinates η and y, as for a plane wave (beam, ray). With the notations used before, the Fourier transform of this external field is given by

$$E_0(\omega, \mathbf{k}; z) = E_0(\omega, \mathbf{k})e^{i\kappa z} \ , \tag{38}$$

where

$$E_0(\omega, \mathbf{k}) = (2\pi)^2 E_0\left[e^{-i(\omega_0-\omega)d/2c}\frac{\sin(\omega_0-\omega)d/2c}{\omega_0-\omega} + (\omega_0 \rightarrow -\omega_0)\right]\cdot$$
$$\cdot\delta(k_x - \tfrac{\omega}{c}\sin\alpha)\delta(k_y) \tag{39}$$

and $\kappa = (\omega/c)\cos\alpha$. The field has the components $E_{0x} = E_0\cos\alpha$, $E_{0z} = -E_0\sin\alpha$ and E_{0y} for the s-wave. We are in the situation described before, where the external field is a transverse monochromatic plane wave, with the only difference that here we have many frequencies ω, each with its corresponding wavevector $\mathbf{K} = (\mathbf{k}, \kappa)$, $\omega = cK$; in particular, $k_y = 0$. We can apply equations (29) given before:

$$E_{t1}(\omega; \mathbf{k}, z) = \tfrac{2\kappa'\lambda^2}{(\kappa'+\kappa)(\kappa\kappa'+k^2)}E_{01}(\omega, \mathbf{k})e^{i\kappa' z} = \tfrac{2\kappa'}{\kappa'+\varepsilon\kappa}E_{01}(\omega, \mathbf{k})e^{i\kappa' z} \ ,$$
$$E_{t2}(\omega; \mathbf{k}, z) = \tfrac{2\kappa}{\kappa'+\kappa}E_{02}(\omega, \mathbf{k})e^{i\kappa' z} \ , \tag{40}$$
$$E_{tz}(\omega; \mathbf{k}, z) = -\tfrac{2k\lambda^2}{(\kappa'+\kappa)(\kappa\kappa'+k^2)}E_{01}(\omega, \mathbf{k})e^{i\kappa' z} = -\tfrac{2k}{\kappa'+\varepsilon\kappa}E_{01}(\omega, \mathbf{k})e^{i\kappa' z} \ ,$$

where $\kappa' = \sqrt{\kappa^2 - \lambda^2\omega_p^2/(\omega^2 - \omega_c^2)} = \sqrt{\varepsilon\lambda^2 - k^2}$. This is the (total) electric field inside the semi-infinite body (the refracted field). According to equations (39) we may put $k_y = 0$, so

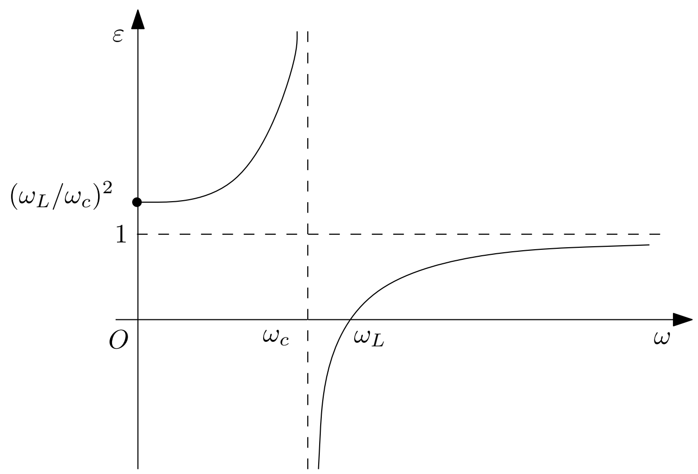

Figure 3: Dielectric function ε vs frequency ω, as given by equation (3).

that $E_{01} = E_{0x}$ and $E_{02} = E_{0y}$. We are interested in the field in the direct space, *i.e.* in the inverse Fourier transforms of equations (40). The integrations with respect to $\mathbf{k}$ can be performed straightforwardly, due to the presence of the δ-functions in equations (40). We get, for instance,

$$E_{t1}(\omega;\mathbf{R}) = E_{01}(\omega)\frac{2\kappa'}{\kappa'+\varepsilon\kappa}e^{i\frac{\omega}{c}\left[x\sin\alpha+z\sqrt{\varepsilon-\sin^2\alpha}\right]} \ , \tag{41}$$

where

$$E_{01}(\omega) = E_{01}\left[e^{-i(\omega_0-\omega)d/2c}\frac{\sin(\omega_0-\omega)d/2c}{\omega_0-\omega} + (\omega_0 \to -\omega_0)\right] \ , \tag{42}$$

$\kappa' = \lambda\sqrt{\varepsilon-\sin^2\alpha}$ and E_{01} is in fact E_{0x} (and E_{t1} is E_{tx}); and similar expressions for the other components of the field. We note first that the refracted field propagates along the refraction direction (it does not propagate anymore along the original, incidence ξ-direction), as expected. In general, this direction depends on ω (through $\varepsilon(\omega)$ in the expression of κ'). We are now left with performing the integral

$$E_{tx}(t,\mathbf{R}) = \frac{1}{2\pi}\int d\omega E_{0x}(\omega)\frac{2\kappa'}{\kappa'+\varepsilon\kappa}e^{i\frac{\omega}{c}\left[x\sin\alpha+z\sqrt{\varepsilon-\sin^2\alpha}-ct\right]} \ . \tag{43}$$

The main contribution to $E_{0x}(\omega)$ (equation (42)) comes from $\omega \simeq \omega_0$, over an interval $\Delta\omega \sim c/d$; it is worth noting that there is no singularity in the wave amplitude coming from the zeros of the denominator $\kappa'+\varepsilon\kappa$; therefore, the polarization eigenmodes do not drive the propagation (as they do in the case of infinite matter). For $\omega_0 \ll \omega_c$, the dielectric function given in Fig. 3 has a slow variation; the integration in equation (43) gives a waves group propagating along the refraction direction and of spatial extension $\simeq c/\Delta\omega = d$, *i.e.* the original pulse preserves approximately its shape and is refracted. The same happens also for $\omega_0 \gg \omega_c$, except that the propagation direction is the original one, as expected ($\varepsilon \to 1$). For ω_0 near ω_c there is a great dispersion, and, since $\varepsilon \to \infty$, the phase factors in equation (43) cancel out; the transmitted field is vanishing; while in the region of anomalous dispersion $\omega_c < \omega < \omega_L$, where $\varepsilon < 0$, we have a great absorption. A similar discussion can be done for the reflected field.

Narrow pulse. We use now the equation of motion (2) (with $\gamma = 0$) with $E_{1,2,z}$ given by equations (17) for u_2 and the combinations $iku_1 + \partial u_z/\partial z$ and $k\partial u_1/\partial z + i\kappa^2 u_z$ in the region $z > 0$. The external field is the pulse given by equation (9). The calculations are straightforward. We get

$$u_1 = Ae^{i\kappa' z} + A_1 e^{i\lambda z} \ , \ u_2 = Be^{i\kappa' z} + B_1 e^{i\lambda z} \ ,$$
$$u_z = -\tfrac{k}{\kappa'} Ae^{i\kappa' z} + \tfrac{\lambda\omega_p^2}{k(\omega^2-\omega_c^2)} A_1 e^{i\lambda z} \ , \tag{44}$$

where

$$A = \tfrac{\lambda^2\omega_p^2}{k^2} \cdot \tfrac{\lambda+\kappa}{\kappa'+\kappa} \cdot \tfrac{\kappa'[\kappa(\omega^2-\omega_c^2)-\lambda\omega_p^2]}{(\kappa\kappa'+k^2)(\omega^2-\omega_c^2)^2} A_1 \ ,$$
$$A_1 = -\tfrac{q}{m} \cdot \tfrac{\omega^2-\omega_c^2}{\omega^2-\omega_c^2-\omega_p^2} \cdot \tfrac{k^2}{(\omega^2-\omega_c^2)k^2+\lambda^2\omega_p^2} E_{01}(\omega) d_t^2 \ , \tag{45}$$

$$B = \tfrac{\lambda^2\omega_p^2}{k^2(\omega^2-\omega_c^2)} \cdot \tfrac{\lambda+\kappa}{\kappa'+\kappa} B_1 \ ,$$
$$B_1 = -\tfrac{q}{m} \cdot \tfrac{k^2}{(\omega^2-\omega_c^2)k^2+\lambda^2\omega_p^2} E_{02}(\omega) d_t^2 \tag{46}$$

and

$$\kappa'^2 = \kappa^2 - \frac{\lambda^2\omega_p^2}{\omega^2-\omega_c^2} = \varepsilon\lambda^2 - k^2 \ . \tag{47}$$

By equation (2) the total electric field $\mathbf{E}_t = \mathbf{E} + \mathbf{E}_0$ (transmitted field) is given by $(\omega^2 - \omega_c^2)\mathbf{u} = -(q/m)\mathbf{E}_t$.

We note that the pulse propagating in matter consists of two distinct parts: 1) the original part of the external pulse proportional to $e^{i\lambda z}$, with a modified amplitude, corresponding to coefficients A_1 and B_1 in equations (44) and 2) an electromagnetic field proportional to $e^{i\kappa' z}$, corresponding to the polaritonic eigenmodes given by $\varepsilon\lambda^2 = \kappa'^2 + k^2$ in equation (47). In addition, we note that, in contrast with the external pulse which is transverse, the field propagating in matter has a longitudinal component corresponding to a non-vanishing u_z in equations (44). This longitudinal component arises from the presence of the surface.

The results given above can be specialized to an external monochromatic plane wave, corresponding to a single frequency ω and a single wavevector $\mathbf{K} = (\mathbf{k}, \kappa)$ (as we did in the preceding sections). In this case, since the external field is transverse, the displacement field given by equations (44) has only the polaritonic contribution $\sim e^{i\kappa' z}$; we can check easily the refraction law $\sin r/\sin i = \omega/cK' = 1/\sqrt{\varepsilon(\omega)}$, where r is the refraction angle, i is the incidence angle and $\mathbf{K}' = (\mathbf{k}, \kappa')$ ($\kappa' = \sqrt{\varepsilon\lambda^2 - k^2}$). A superposition of such waves with different frequencies ω and is a beam (ray) of finite duration and spatial extension along its propagation direction; according to the above results, the refraction of such a pulse exhibits dispersion.

We are interested now in taking the inverse Fourier transforms of equations (44) in order to get the space and time dependence of the field. We notice now that, in contrast with the external pulse, we have a transverse dispersion, through the k-dependence. We focus first on the angular integration in the Fourier transform with respect to $\mathbf{k}$, taking advantage of the fact that the field depends only on the magnitude k. In the partial Fourier transform

$$\mathbf{u}(\omega; \mathbf{r}, z) = \frac{1}{(2\pi)^2} \int d\mathbf{k}\mathbf{u}(\omega, \mathbf{k}; z) e^{i\mathbf{k}\mathbf{r}} \tag{48}$$

we take $\mathbf{r} = r(\cos\theta, \sin\theta)$, $\mathbf{k} = k(\cos\varphi, \sin\varphi)$ and $\mathbf{k}_\perp = k(-\sin\varphi, \cos\varphi)$ and integrate over angle φ. We get easily that $u_{1,2}$ corresponds to the radial and, respectively, tangential components of the field,

$$u_{r,\theta}(\omega; \mathbf{r}, z) = \frac{i}{2\pi} \int_0 dk \cdot k u_{1,2}(\omega, \mathbf{k}; z) J_1(kr) \ , \tag{49}$$

and

$$u_z(\omega;\mathbf{r},z)=\frac{1}{2\pi}\int_0 dk\cdot ku_z(\omega,\mathbf{k};z)J_0(kr)\ \ , \tag{50}$$

where $J_{0,1}$ are the Bessel functions of the zeroth and, respectively, first order.

1) Large transverse distance. It is easy to see from equations (49) and (50) that for large distances r the field goes like $1/r^2$, as expected for a localized electromagnetic perturbation. In this case only the $k\simeq 0$ components contribute to the integrals. Taking the limit $k\rightarrow 0$ in equations (44)-(47) we see that the original, external pulse is lost in this case (A_1, $B_1\rightarrow 0$), as well as the longitudinal component u_z, which is vanishing in this limit. For large transverse distances the propagation is governed by the polaritonic eigenmodes (with the longitudinal wavevector $\kappa'\simeq\lambda\sqrt{\varepsilon}$), as expected. It is convenient to give the expression of the total electric field

$$\mathbf{E}_t(\omega;\mathbf{r},z)\simeq\frac{1}{\sqrt{\varepsilon}+1}\mathbf{E}_0(\omega)\frac{d_t^2}{r^2}e^{i\lambda\sqrt{\varepsilon}z}\ \ ,\ r\gg d_t\ . \tag{51}$$

The inverse temporal Fourier of equation (51) proceeds in a similar manner as for the external pulse. First, we note that $\kappa'\simeq\lambda\sqrt{\varepsilon}$ may have purely imaginary values, corresponding to the band $\omega_c<\omega<\omega_c+\omega_p$ of anomalous dispersion. In this case, the field is damped with respect to the longitudinal coordinate z. In this respect, it is worth noting that κ' arises from $\varepsilon\lambda^2=\kappa'^2$, and λ in equation (51) is in fact $|\lambda|$; consequently, a corresponding sign should be attached to $\sqrt{\varepsilon}$ when it is purely imaginary. More convenient is to use the expression given by equation (51) for ω near ω_0 and to add the complex conjugate (corresponding to ω near $-\omega_0$). Leaving aside the damped mode, the inverse Fourier transform in equation (51) gives a pulse of, practically, the same length d as the original, external pulse, propagating with the group velocity $v_g=c/[\partial(\lambda\sqrt{\varepsilon})/\partial\omega]_{\omega=\omega_0}$, or the more familiar form $v_g=\partial(\omega/\partial\kappa')_{\kappa'=\kappa_0'}$, where $\sqrt{\varepsilon}\lambda=\kappa'$ for $\omega=\omega_0$ and $\kappa'=\kappa_0'$. This group velocity is always smaller than the speed of light. As usually, if the dispersion is too high (as for instance for a temporal δ-pulse), a group propagation cannot be defined any longer.

2) Small transverse distance. In the region of small tranverse distances the Fourier components with large wavevectors k contribute to the integrals in equations (49) and (50). We take the limit $k\rightarrow\infty$ in equations (44)-(47) and get

$$E_{tr}(\omega;\mathbf{r},z)\simeq\tfrac{1}{\varepsilon}E_{0r}(\omega)e^{i\lambda z}\ \ ,\ \ E_{t\theta}(\omega;\mathbf{r},z)\simeq E_{0\theta}(\omega)e^{i\lambda z}\ \ ,\ r\simeq d_t\ . \tag{52}$$

Therefore, in the central region $r\simeq d_t\rightarrow 0$ the original, external pulse penetrates into the body practically with the same shape and the amplitude affected by the dielectric function $\varepsilon(\omega_0)$. The motion of the pulse in this region is not affected by the polaritonic eigenmodes, which give only a damped field; it is governed mainly by the external pulse; it propagates with the speed of light c. It is reminiscent of the propagation of the front of the signal.[1]-[4] However, it is worth noting that the radial component of the field is changed. We can compute the (average) density of energy trasmitted by this pulse, $u_t\simeq E_0^2(1+1/2\varepsilon^2)/4\pi$ (the radial component of the electric field has not an associated magnetic field), which should be compared with the incident density of energy $E_0^2/8\pi$ (the same ralationship holds also for energy flows). We get a transmission coefficient $T\simeq[1+1/2\varepsilon^2(\omega_0)]/2$, which is valid for $\varepsilon^2>1/2$. In the opposite case, where ε is vanishing, we are approaching a resonant regime corresponding to the excitation of the longitudinal mode ω_L.

Reflected pulse. Equations (17) give also the reflected field for $z<0$. Making use of the displacement field given by equations (44) we get

$$\begin{aligned}E_1=2\pi\{&\tfrac{\lambda^2\omega_p^2}{k^2}\cdot\tfrac{(\lambda+\kappa)(\kappa\kappa'-k^2)}{(\kappa'+\kappa)^2(\kappa\kappa'+k^2)}\cdot\tfrac{\kappa(\omega^2-\omega_c^2)-\lambda\omega_p^2}{(\omega^2-\omega_c^2)^2}+\\&+\tfrac{\kappa(\omega^2-\omega_c^2)+\lambda\omega_p^2}{(\lambda+\kappa)(\omega^2-\omega_c^2)}\}A_1e^{-i\kappa z}\end{aligned} \tag{53}$$

$E_z = kE_1/\kappa$ and

$$E_2 = \frac{2\pi\lambda^2}{\kappa}\left[\frac{\lambda^2\omega_p^2}{k^2(\omega^2-\omega_c^2)}\cdot\frac{\lambda+\kappa}{(\kappa'+\kappa)^2}+\frac{1}{\lambda+\kappa}\right]B_1 e^{-i\kappa z} \ , \tag{54}$$

where A_1 and B_1 are given by equations (45) and (46). We note that that this is indeed a reflected field ($\sim e^{-i\kappa z}$), which is transverse ($div\mathbf{E} = 0$ since $kE_1 - \kappa E_z = 0$), as expected. The analysis goes in the same manner as for the transmitted field. At large transverse distances we get

$$\mathbf{E}(\omega; \mathbf{r}, z) \simeq -\frac{\omega_p^2}{(\sqrt{\varepsilon}+1)^2(\omega^2-\omega_c^2)}\mathbf{E}_0(\omega)\frac{d_t^2}{r^2}e^{-i\lambda z} \ , \ r \gg d_t \ , \tag{55}$$

while for small transverse distances the field is damped (it goes like e^{kz} for $z < 0$).

Therefore, we can say that the external pulse penetrates almost entirely in the body in the region of small transverse distances where it is localized, while the body responds through its polaritonic eigenmodes both in the transmitted and reflected field at large transverse distances. Part of the pulse energy is transmitted into the small transverse distance region, according to the transmission coefficient given above, another part is transmitted and reflected through the oscillations of the polaritonic eigenmodes in the region of larger transverse distances; and another part of the energy is transferred to the mechanical energy of the polarization degrees of freedom. Indeed, it is worth noting in this context that Maxwell equations (1) (for the total field) and the equation of motion (2) leads to the law

$$\frac{\partial}{\partial}\left[\frac{1}{8\pi}(E_t^2+H_t^2)+\frac{1}{2}nm(\dot{\mathbf{u}}^2+\omega_c^2\mathbf{u}^2)\right]+\frac{c}{4\pi}div(\mathbf{E}_t\times\mathbf{H}_t) = 0 \tag{56}$$

of energy conservation. We can see the mechanical energy of the mobile charges in equation (56), whose variation is equal to the mechanical work done by the field upon the mobile charges.

Oblique incidence. We refer to the coordinates introduced by equations and consider a pulse of the from

$$E_0(t, \mathbf{R}) = E_{0\eta}(t, \mathbf{R}) = E_0\cos\omega_0(t-\xi/c)\theta(ct-\xi)\theta(\xi-ct+d)\cdot d_t^2\delta(\eta)\delta(y) \ ; \tag{57}$$

equation (57) describes a pulse of zero thickness and length d, propagating in vacuum along the ξ-direction which makes an incidence angle α with a plane surface placed at $z = 0$; the main frequency is ω_0 and the electric field is directed along the transverse η-axis, which is perpendicular to the y-axis and the direction of propagation ξ; the associated magnetic field is directed along the y-axis.

The Fourier transform of this field is given by

$$E_{0x,z}(\omega, \mathbf{k}; z) = E_{0x,z}(\omega)e^{i\overline{\lambda}z} \ , \tag{58}$$

where

$$E_{0x}(\omega) = E_0(\omega)d_t^2 \ , \ E_{0z}(\omega) = -E_0(\omega)d_t^2\tan\alpha \ , \tag{59}$$

$$\overline{\lambda} = (\lambda - k_x\sin\alpha)/\cos\alpha \tag{60}$$

and $E_0(\omega)$ is given by equation (10). We use this external field in equations (17) and introduce the internal (polarization) field obtained this way in the equation of motion (2) to get the displacement field $\mathbf{u}$ and, especially, the total electric field $\mathbf{E}_t$ inside the body. Making use of the displacement field $\mathbf{u}$ in equations (17) we get also the reflected field for $z<0$. The calculations are straightforward and go in the same manner as before. We get

$$u_1 = Ae^{i\kappa' z} + A_1 e^{i\overline{\lambda} z} \ , \ u_2 = Be^{i\kappa' z} + B_1 e^{i\overline{\lambda} z} \ ,$$
$$u_z = -\tfrac{k}{\kappa'} Ae^{i\kappa' z} + \tfrac{k\overline{\lambda}}{\kappa'^2} Ce^{i\overline{\lambda} z} \ , \tag{61}$$

where

$$A = -\tfrac{\kappa\kappa'}{\kappa\kappa' + k^2} \cdot \tfrac{\kappa' - \kappa}{\overline{\lambda} - \kappa} \left(A_1 + \tfrac{k^2 \overline{\lambda}}{\kappa\kappa'^2} C \right) \ ,$$
$$A_1 = -\tfrac{q}{m} \cdot \tfrac{1}{\omega^2 - \omega_c^2 - \omega_p^2} \cdot \tfrac{(\varepsilon\overline{\lambda}^2 - \kappa'^2) E_{01}(\omega) + (\varepsilon\kappa^2 \overline{\lambda}/k - \kappa'^2) E_{0z}(\omega)}{\overline{\lambda}^2 - \kappa'^2} \ , \tag{62}$$

$$B = -\tfrac{\kappa' - \kappa}{\overline{\lambda} - \kappa} B_1 \ ,$$
$$B_1 = -\tfrac{q}{m} \cdot \tfrac{1}{(\omega^2 - \omega_c^2)} \tfrac{\overline{\lambda}^2 - \kappa^2}{\overline{\lambda}^2 - \kappa'^2} E_{02}(\omega) \tag{63}$$

and

$$C = -A_1 - \frac{q}{m} \frac{E_{01}(\omega) + (\kappa^2/k\overline{\lambda}) E_{0z}(\omega)}{\omega^2 - \omega_c^2} \ . \tag{64}$$

We are interested in inverse Fourier transforms of the type given by equation (48). We note that $\mathbf{u}(\omega, \mathbf{k}; z)$ given by equations (61) are of the form

$$\mathbf{u}(\omega, \mathbf{k}; z) = \mathbf{u}_e(\omega, \mathbf{k}) e^{i\kappa' z} + \mathbf{u}_p(\omega, \mathbf{k}) e^{i\overline{\lambda} z} \ , \tag{65}$$

where $\mathbf{u}_{e,p}(\omega, \mathbf{k})$ are contributions coming from the polaritonic eigenmodes (wavevector κ') and, respectively, the pulse (wavevector $\overline{\lambda}$) in equations (61)-(64). It is convenient to introduce the coordinates ξ and η and write the Fourier transform as

$$\mathbf{u}(\omega; \mathbf{r}, z) = \tfrac{1}{(2\pi)^2} \int dk_x dk_y \cdot$$
$$\cdot \{ \mathbf{u}_e(\omega, \mathbf{k}) e^{i(\kappa' \cos\alpha + k_x \sin\alpha)\xi} e^{i(-\kappa' \sin\alpha + k_x \cos\alpha)\eta} +$$
$$+ \mathbf{u}_p(\omega, \mathbf{k}) e^{i\lambda\xi} e^{i(-\lambda \tan\alpha + k_x/\cos\alpha)\eta} \} e^{ik_y y} \ . \tag{66}$$

We can see from equation (66) that for small distances along the y- and η-directions it is sufficient to retain the contributions coming from large values of the transverse wavevector $\mathbf{k}$. It is easy to see in this case that κ and κ' are almost equal to each other (approximately ik). Consequently, the amplitude coefficients A and B given by equations (62) and (63) are vanishing, since they contain the factor $\kappa' - \kappa$. This means that the polaritonic eigenmodes do not contribute, while the external, original pulse preserves its shape and propagation direction, as expected. However, the amplitude of the pulse, governed by the coefficients A_1,B_1 and C, is changed. Although this amplitude depends on the position in a complicated way, it is easy to see that it does not vary much in the neigbourhood of the propagation direction, where the field can be approximated by equation (52), as for normal incidence.

For large distances along the transverse y- and η-directions, we notice that the contributions arising from $k_y \to 0$ and $k_x \to \lambda \sin\alpha$ bring the main contribution to the integral in equation (66) (at first sight we should consider also the case $k_x \to \varepsilon\lambda \sin\alpha$, but the corresponding amplitudes are much smaller in this case). Under these circumstances we have $\overline{\lambda} \simeq \kappa \simeq \lambda \cos\alpha$, so that the leading contributions arise from the coefficients A and B in equations (62) and (63), corresponding to the polaritonic eigenmodes, as expected (note the factor $\overline{\lambda}-\kappa$ in the denominator of these coefficients in equations (62) and (63)). We may say, in conclusion, that the main features of the propagation of the pulse at oblique incidence are the same as for normal incidence.

The reflected field can be computed from equations (17) for $z<0$, making use of the displacement field given by equations (61). For both the eingenmodes and pulse contributions we get a reflection factor $e^{-i\kappa z}$ in the field. For small transverse distances the transverse wavevector $\mathbf{k}$ goes to infinity and, consequently, the reflected field is damped *i.e.* the original pulse penetrates practically in its entirety into the body. At large distance from the pulse the reflected field is controlled by the polaritonic response of the body. We note that a change $z\rightarrow -z$ in equations (57) corresponds indeed to a reflection of the coordinates ξ and η with respect to the plane $z=0$.

Concluding remarks. The propagation of an electromagnetic pulse in a semi-infinite body (half-space) was investigated here, taking into acount the dispersion caused by the body polarization. The pulse considered in this investigation was of finite duration and of finite spatial extension along its propagation direction. Along the transverse directions we considered two cases: a large (plane wave, beam, ray) extension and a vanishing thickness (narrow δ-pulse along transverse directions). For a pulsed plane wave we found that the law of refraction is respected. For a narrow pulse the propagation was considered both at normal and oblique incidence. It was shown that the pulse preserves approximately its shape and propagation velocity (the speed of light in vacuum) in the body, with a distorted amplitude controlled by the dielectric function, while, at large distances from the propagation direction the pulse produces a disturbance which is governed by the polaritonic eigenmodes. This disturbance extends over all the space along the transverse directions, vanishing rapidly at infinity (as the inverse square of the distance), has a finite extension along the propagation direction and propagates with the group velocity corresponding to the polaritonic eigenmodes.

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