Journal of Theoretical Physics

Founded and Edited by M. Apostol **213** (2012)

ISSN 1453-4428

Reading Brillouin

M. Apostol
Department of Theoretical Physics,
Institute of Atomic Physics, Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

In two classical papers[1, 2] Sommerfeld and Brillouin examined the propagation velocity of the electromagnetic radiation in matter (see also Refs. [3, 4]). Matter is known to be dispersive. This investigation was motivated by phase velocities which exceed the speed of light in vacuum c, in relation to the Theory of Relativity.

From ancient times it is known from experiment the Snell's law refraction law $\sin r/\sin i = 1/n$, where r is the refraction angle, i is the incidence angle and n is the refraction index. It is known from Huygens theory the law $\sin r/\sin i = v_\phi/c = \omega/ck$, where ω is the frequency and $\mathbf{k}$ is the wavevector of a monochromatic plane wave with the phase velocity $v_\phi = \omega/k$. From Maxwell equations it is also known $\varepsilon\omega^2 = c^2k^2$ (for non-magnetic matter), where ε is the dielectric constant (or $\varepsilon\mu\omega^2 = c^2k^2$ for magnetic matter, where μ is the magnetic permeability). The Snell's law of refraction reads

$$\frac{\sin r}{\sin i} = \frac{v_\phi}{c} = \frac{\omega}{ck} = \frac{1}{\sqrt{\varepsilon}} = \frac{1}{n} \ . \tag{1}$$

It is known from experiment that ε, v_ϕ, k, ε and n depend all on the frequency ω. For some frequencies, v_ϕ may trespass the speed of light c as, for instance, near a vanishing ε. The situation is further complicated by the anomalous dispersion region, where ε is negative (actually, the relation $\varepsilon\omega^2 = c^2k^2$ must be satisfied with $\varepsilon > 0$, so the expression of the Snell's law with ε has a limited validity). The nature of the refraction law, as well as many others aspects related to the propagation of the electromagnetic waves in matter has been clarified recently to a large extent by the theory of polarizable matter and polaritonic eigenmodes. However, the problem of superluminic phase velocities persists.

In this context, it is worth recalling that the theory of polarizable matter and plasmonic and polaritonic eigenmodes is based on the Drude-Lorentz (plasma) model.[5]-[7] It is also known of long ago the so-called extinction theorem, which tells that free electromagnetic waves, *i.e.* electromagnetic waves propagating with the speed of light in vacuum c, cannot be propagated in matter.[8]-[10]

The propagation of the electromagnetic waves in matter implies both a separation surface of the body from the vacuum (or an interface between two bodies) and a signal of a finite duration (and finite spatial extension). The presence of the surface leads to refraction and the lack of translational symmetry, which complicates the analysis. A finite duration and a finite spatial extension imply a superposition of frequencies and wavevectors, related through the phase velocity. In a simplifying approach the analysis of Sommerfeld and Brillouin leads to an electromagnetic signal represented by

$$f(x,t) = \int dk e^{-i\Omega(k)t+ikx} \ \ , \tag{2}$$

where $\Omega(k)$ is the polaritonic frequency. The body is assumed to be infinite, the extension of the wave along the transverse directions is supposed to be infinite (as for a plane wave, or a beam, ray in geometrical optics, in order to leave only one component of the wavevector). Typical polaritonic frequencies are given by $\Omega(k)=\sqrt{\omega_p^2+c^2k^2}$ (e.g. for conductors) and $\Omega(k)=vk\omega_T/(vk+\omega_T)$ for dielectrics, where $\omega_T$is the so-called transverse-mode frequency and $v=c\omega_T/\omega_L(<c)$, where $\omega_L$is the so-called frequency o the longitudianl modes ($\omega_L=\sqrt{\omega_p^2+\omega_T^2}$).

The analysis o Brillouin and Sommerfeld proeceds as follows. We divide the whole domain of integration in equation (2) ($-\infty$ to $+\infty$) by some intervals Δ_i around some points k_i, such as within each interval Δ_i we may approximate the frequency by

$$\Omega(k)=\Omega(k_i)+v_i(k-k_i)+\frac{1}{2}\Omega^{''}(k_i)(k-k_i)^2+..., \tag{3}$$

where $v_i=\Omega^{'}(k_i)=\partial\Omega/\partial k|_{k=k_i}$. The integral in equation (2) becomes

$$f(x,t)\simeq\sum_i e^{-i\Omega_i t+ik_ix}\int_{\Delta_i}dqe^{i(x-v_it)q}e^{-\frac{1}{2}i\Omega_i^{''}tq^2}\ \ , \tag{4}$$

where $\Omega_i=\Omega(k_i)$, $\Omega_i^{''}=\Omega^{''}(k_i)$ and $q=k-k_i$ within each interval Δ_i. If we may neglect the second-order derivative and extend the integration over sufficiently large intervals, we can see that we get pulses of the form $\delta(x-v_it)$ propagating with velocity v_i. This is called the group velocity. The second-order derivative leads to an (approximate) Fresnel integral and makes the pulses flat, with a spread increasing with increasing the time, according to an imaginary Gauss function (as in an imaginary diffusion). Moreover, making use of the form of the polaritonic frequency $\Omega(k)$ we can see that the group velocity is always smaller than the speed of light in vacuum c, and for high frequency it approaches c, *i.e.* the front of the wave moves almost with the speed of light c in matter. Therefore, the wave in dispersive matter moves as a set of groups, each with a group velocity, getting flat in time; if there is one main group, there are others moving with higher velocity (but smaller than the sped of light in vacum), which we call precursors (or forerunners), beside others which lag behind the main group. This is the group-velocity picture based on the method of stationary phase (steepest descent or saddle point method).[11]

Still, there is a discussion regarding superluminal velocities of electromagnetic signals.

Indeed, the picture described above is based on an approximation, concerning the extension of the integration interval Δ_i to, practically, infinity. Actually, the integration over a finite interval Δ_i leads to a wavepacket, of the form

$$\int_{\Delta_i}dqe^{i(x-v_it)q}=\frac{2\sin(x-v_it)\Delta_i/2}{x-v_it}\ \ , \tag{5}$$

which, it is true, tends to $\pi\delta(x-v_it)$ for $\Delta_i\rightarrow\infty$, but it extends to infinite $x-v_it$ for a finite interval Δ_i. This means that each wave group extends instantaneously to infinity (of course, with a decreasing amplitude), which suggests indeed a velocity much higher than the speed of light in vacuum (practically an infinite one). This feature is due to the non-locality of the waves, and it implies in fact the "propagation" of no signal.

The non-local picture of the waves can be seen in fact more directly on the integral in equation (2). At $t=0$ we have

$$f(x,t=0)=\int dke^{-ikx}=2\pi\delta(x)\ \ , \tag{6}$$

and the signal is highly localized (on $x=0$); at the next infinitesimal moment of time Δt we have

$$f(x,\Delta t)=\int dke^{-i\Omega(k)\Delta t+ikx}\ \ , \tag{7}$$

and the signal is already extended over the whole space; since $x/\Delta t \to \infty$ for any x and $\Delta t \to 0$ we may say that the signal has been propagated indeed with an infinite velocity.

References

[1] A. Sommerfeld, "Uber die Fortpflanzung des Lichtes in dispergierenden Medien", Ann. Phys. **44** 177–202 (1914).

[2] L. Brillouin, "Uber die Fortpflanzung des Lichtes in dispergierenden Medien", Ann. Phys. **44** 203-240 (1914).

[3] L. Brillouin, *Wave Propagation and Group Velocity*, Academic Press, NY (1960).

[4] A. Sommerfeld, *Vorlesungen uber Theoretische Physik*, Band IV, (*Optik*), Akademische Verlag, Leipzig (1964).

[5] P. Drude, "Zur Elektronentheorie der Metalle", Ann. Phys. **306** 566-613 (1900).

[6] P. Drude, "Zur Elektronentheorie der Metalle, 2. Teile. Galvanomagnetische und thermomagnetische Effecte", **308** 369-402 (1900).

[7] H. A. Lorentz, *The Theory of Electrons*, Teubner, Leipzig (1916).

[8] P. P. Ewald, "Uber die Grundlagen der Kristalloptik", Thesis, Munich (1912), Ann. Phys. **49** 1- (1916).

[9] C. W. Oseen, "Uber die Wechselwirkung zwischen zwei elektrischen Dipolen der Polarisationsebene in Kristallen und Fluessigkeiten", Ann. Phys. **48** 1-56 (1915).

[10] M. Born and E. Wolf, *Principles of Optics*, Pergamon, London (1959).

[11] P. Debye, "Näherungsformeln für die Zylinderfunktionen für große Werte des Arguments und unbeschränkt veränderliche Werte des Index", Mathematische Annalen **67** 535–558 (1909).