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A resonant coupling of a localized harmonic oscillator to an elastic medium

B. F. Apostol
Department of Seismology, Institute for Earth's Physics,
Magurele-Bucharest Mg-6, POBox Mg-35, Romania
email: afelix@theory.nipne.ro

Abstract

The motion is analyzed of a point harmonic oscillator coupled to a homogeneous elastic medium and localized either on the medium surface or embedded in the medium. Two new elements are introduced, one regarding the reaction of the oscillator to the medium and another a coupling factor. The present analysis is meant to be relevant for the effects a seismic motion may have upon a localized struture, either natural or man-built. It is shown that the reaction of the oscillator modifies its inertia, which in turn leads to a change in the oscillator's eigenfrequency; this change is controlled by the coupling function. The present treatment opens the way to introduce new, more realistic features in analyzing the effect of the seismic motion uponlocalzeid structures, in particular the non-linear features of the coupling of the structure with its local site motion.

Introduction. In studies of seismic risk and hazard it is of utmost importance to assess the effects of the seismic motion upon localized structures, either natural or man-built. Usually, such structures are viewed as localized harmonic oscillators, with one or several degrees of freedom and corresponding eigenfrequencies (characteristic frequencies). It is assumed that the seismic motion acts as an external force upon such oscillators and the resonant regime is highlighted. It is desirable of course to avoid the resonance, *i.e.* the structure's characteristic frequencies must be different from the main frequencies of the seismic motion at the site of the structure (local seismic motion).

Two essential elements are overlooked in such a simplified picture: the reaction of the structure back on the elastic medium and the coupling of the structure to the elastic medium. We show here a way of introducing these two elemnts in the analysis and describe the consequences, some surprising, of including these two more realistic features.

Structure on the surface. First, we consider the free plane surface of an infinite, homogeneous elastic medium; we consider elastic waves propagating on this surface (Rayleigh waves) and assume a generic wave equation

$$\rho\ddot{\mathbf{u}} = F\Delta\mathbf{u} \tag{1}$$

describing the motion of the (two-dimensional) displacement vector $\mathbf{u}$; in equation (1) ρ is the superficial mass density and F is a generic superficial modulus of elasticity, such that the wave velocity is given by $c^2 = F/\rho$; the laplacian in equation (1) is the two-dimensional laplacian. On the other hand we assume a point-like harmonic oscillator with mass m and eigenfrequency Ω localized at $\mathbf{r}_0$ on the surface, described by the equation

$$m\ddot{\mathbf{v}} + m\Omega^2\mathbf{v} = 0 \ , \tag{2}$$

where $\mathbf{v}$ is the oscillator's displacement from its equilibrium position.

The medium acts upon the oscillator with the force superficial density $(F\Delta\mathbf{u})_{\mathbf{r}=\mathbf{r}_0}$; for an area S of the contact surface between the oscillator and the medium, the force acting upon the oscillator is $S(F\Delta\mathbf{u})_{\mathbf{r}=\mathbf{r}_0}$. Here we introduce a coupling function g and write the force as $gS(F\Delta\mathbf{u})_{\mathbf{r}=\mathbf{r}_0}$; under the conditions, the equation of motion of the oscillator becomes

$$m\ddot{\mathbf{v}} + m\Omega^2\mathbf{v} = gS(F\Delta\mathbf{u})_{\mathbf{r}=\mathbf{r}_0} \; ; \tag{3}$$

the area S must be much smaller than the area constructed with any relevant wavelength. The coupling function g may have a complex structure; it may depend on the oscillator eigenmode (frequency Ω), on the oscillator amplitude, on the local amplitude $\mathbf{u}$ of the wave and even on the time t. We assume here the most simple situation which corresponds to a constant g. Obviously, $g \leq 1$.

Similarly, the oscillator reacts back upon the elastic medium, with its inertia force $-gm\ddot{\mathbf{v}}S\delta(\mathbf{r}-\mathbf{r}_0)$, localized at $\mathbf{r}_0$ and affected by the coupling function; the wave equation (1) becomes

$$\rho\ddot{\mathbf{u}} = F\Delta\mathbf{u} - gm\ddot{\mathbf{v}}\delta(\mathbf{r}-\mathbf{r}_0) \; . \tag{4}$$

Equations (3) and (4) are two coupled equations, which we solve here. We write equation (4) as

$$\frac{1}{c^2}\ddot{\mathbf{u}} - \Delta\mathbf{u} = -\frac{gm\ddot{\mathbf{v}}}{F}\delta(\mathbf{r}-\mathbf{r}_0) \; , \tag{5}$$

take the temporal Fourier transform, introduce the modulus of the wavevector $k = \omega/c$ and get

$$\Delta\mathbf{u} + k^2\mathbf{u} = -\frac{gm\omega^2\mathbf{v}}{F}\delta(\mathbf{r}-\mathbf{r}_0) \; ; \tag{6}$$

the solution of the equation

$$\Delta u + k^2 u = f \tag{7}$$

in two dimensions is given by[1]

$$u = \frac{1}{4i}\int d\mathbf{r}' H_0^{(1)}(k\left|\mathbf{r}-\mathbf{r}'\right|)f(\mathbf{r}') \; , \tag{8}$$

since

$$\Delta H_0^{(1)} + k^2 H_0^{(1)} = 4i\delta(\mathbf{r}) \; , \tag{9}$$

$H_0^{(1)}$ being the Hankel function of zeroth degree and, at the same time, the Green function in equation (); its asymptotic behaviour is given by

$$H_0^{(1)}(kr) \sim \begin{cases} \frac{2i}{\pi}\ln(kr) \; , \; kr \to 0 \; , \\ \sqrt{\frac{2\pi}{kr}}e^{i(kr+\pi/4)} \; , \; kr \to \infty \; . \end{cases} \tag{10}$$

Applying these formulae to equation (6) we get the particular solution

$$\mathbf{u}_p = -\frac{gm\omega^2\mathbf{v}}{4iF}H_0^{(1)}(k\left|\mathbf{r}-\mathbf{r}_0\right|) \simeq -\frac{gm\omega^2\mathbf{v}}{2\pi F}\ln(k\left|\mathbf{r}-\mathbf{r}_0\right|) \; , \; k\left|\mathbf{r}-\mathbf{r}_0\right| \ll 1 \; ; \tag{11}$$

we can see that a localized source generates cylindrical waves on an elastic surface, which have a logarithmic singularity at the source. A solution of the homogeneous equation (6) must be added to this particular solution in order to get the general solution; we choose a free wave written as

$$\mathbf{u}_0 = \mathbf{A}\cos\omega_0(t - x/c) \; , \tag{12}$$

or its Fourier transform

$$\mathbf{u}_0 = \pi\mathbf{A}\left[\delta(\omega-\omega_0)e^{i\omega_0 x/c} + \delta(\omega+\omega_0)e^{-i\omega_0 x/c}\right] \ . \tag{13}$$

Now we compute $(\Delta\mathbf{u})_{\mathbf{r}=\mathbf{r}_0}$, where $\mathbf{u}=\mathbf{u}_p+\mathbf{u}_0$, in order to introduce it in equation (3); since $\Delta\ln r_{\mathbf{r}=0}=2\pi\delta(\mathbf{r})$ we have

$$(\Delta\mathbf{u}_p)_{\mathbf{r}=\mathbf{r}_0} = -\frac{gm\omega^2\mathbf{v}}{F}\delta(\mathbf{r}-\mathbf{r}_0)_{\mathbf{r}=\mathbf{r}_0} \simeq -\frac{gm\omega^2\mathbf{v}}{FS} \ , \tag{14}$$

while

$$(\Delta\mathbf{u}_p)_{\mathbf{r}=\mathbf{r}_0} = -\frac{\omega_0^2}{c^2}\pi\mathbf{A}\left[\delta(\omega-\omega_0)e^{i\omega_0 x_0/c} + \delta(\omega+\omega_0)e^{-i\omega_0 x_0/c}\right] \ . \tag{15}$$

Introducing these quantities in the Fourier transform of equation (3) we get

$$\begin{gathered} m(\Omega^2-\omega^2)\mathbf{v} = -g^2 m\omega^2\mathbf{v} - \\ -\tfrac{gSF\omega_0^2}{c^2}\pi\mathbf{A}\left[\delta(\omega-\omega_0)e^{i\omega_0 x_0/c} + \delta(\omega+\omega_0)e^{-i\omega_0 x_0/c}\right] \ , \end{gathered} \tag{16}$$

or

$$\begin{gathered} \mathbf{v}(\omega) = -\tfrac{\omega_0^2}{\Omega^2-\omega^2(1-g^2)}\times \\ \times\tfrac{gS\rho}{m}\pi\mathbf{A}\left[\delta(\omega-\omega_0)e^{i\omega_0 x_0/c} + \delta(\omega+\omega_0)e^{-i\omega_0 x_0/c}\right] \ . \end{gathered} \tag{17}$$

The most important result exhibited by equation (17) is the change in the resonance frequency $\omega \rightarrow \omega\sqrt{1-g^2}$. As a result of its interaction with the elastic medium, the oscillator eigenfrequncy Ω changes into $\Omega/\sqrt{1-g^2}$ (gets "renormalized"). If we take the inverse Fourier transform we get

$$v(t) = -\frac{\omega_0^2}{\Omega^2-\omega_0^2(1-g^2)}\frac{gS\rho}{m}\mathbf{A}\cos\omega_0(t-x_0/c) \ ; \tag{18}$$

if we take into accont the contribution of the poles $\omega=\pm\Omega/\sqrt{1-g^2}$ we get the solution corresponding to free oscillations at resonance; which occurs now at the modified eigenfrequency $\pm\Omega/\sqrt{1-g^2}$. It is worth noting that for a perfect coupling corresponding to $g=1$, there is not a resonance anymore.

Point oscillator embedded in an elastic medium. Although not very realistic, we examine here the case of a point oscillator embedded in a homogeneous, infinite, elastic medium because the solution may appear more familiar. Equations (3) and (4) read now

$$\begin{gathered} m\ddot{\mathbf{v}} + m\Omega^2\mathbf{v} = gV(F\Delta\mathbf{u})_{\mathbf{R}=\mathbf{R}_0} \ , \\ \rho\ddot{\mathbf{u}} = F\Delta\mathbf{u} - gm\ddot{\mathbf{v}}\delta(\mathbf{R}-\mathbf{R}_0) \ , \end{gathered} \tag{19}$$

where ρ is the volumic density of mass, F is a generic (volumic) modulus of elasticity; the wave velocity has the same expression (and value) given by $c^2=F/\rho$ and V (much smaller than the volume constructed with any relevant wavelength) is the volume of the oscillator. The wave equation

$$\frac{1}{c^2}\ddot{u} - \Delta u = f \tag{20}$$

has the particular solution[1]

$$u = \frac{1}{4\pi}\int d\mathbf{R}'\frac{f(\mathbf{R}', t-|\mathbf{R}-\mathbf{R}'|/c)}{|\mathbf{R}-\mathbf{R}'|} \ ; \tag{21}$$

applying this formula to the wave equation

$$\frac{1}{c^2}\ddot{\mathbf{u}} - \Delta\mathbf{u} = -\frac{gm\ddot{\mathbf{v}}}{F}\delta(\mathbf{R}-\mathbf{R}_0) \tag{22}$$

we get the particular solution

$$\mathbf{u}_p = -\frac{gm\ddot{\mathbf{v}}(t-|\mathbf{R}-\mathbf{R}_0|/c)}{4\pi F|\mathbf{R}-\mathbf{R}_0|} \tag{23}$$

and

$$(\Delta\mathbf{u}_p)_{\mathbf{R}=\mathbf{R}_0} = \frac{gm\ddot{\mathbf{v}}(t)}{F}\delta(\mathbf{R}-\mathbf{R}_0)_{\mathbf{R}=\mathbf{R}_0} = \frac{gm\ddot{\mathbf{v}}(t)}{FV} \ , \tag{24}$$

since $\Delta(1/R) = -4\pi\delta(\mathbf{R})$. Similarly, for a plane wave $\mathbf{u}_0 = \mathbf{A}\cos\omega_0(t-x/c)$ we get

$$(\Delta\mathbf{u}_0)_{\mathbf{R}=\mathbf{R}_0} = -(\omega_0^2\mathbf{A}/c^2)\cos\omega_0(t-x_0/c) \ . \tag{25}$$

The equation of motion (19) of the oscillator becomes

$$m\ddot{\mathbf{v}} + m\Omega^2\mathbf{v} = g^2 m\ddot{\mathbf{v}} - \frac{gVF\omega_0^2}{c^2}\mathbf{A}\cos\omega_0(t-x_0/c) \ , \tag{26}$$

or

$$m(1-g^2)\ddot{\mathbf{v}} + m\Omega^2\mathbf{v} = -g\rho V\omega_0^2\mathbf{A}\cos\omega_0(t-x_0/c) \ . \tag{27}$$

This is a typical equation of motion for a harmonic oscillator with a modified eigenfrequency under the action of an external force.

Conclusion. In conclusion we may say that the reaction of a point harmonic oscillator to the elastic medium to which it is coupled modifies the inertia of the oscillator, which implies a change in its eigenfrequency; this change is controlled by the coupling function. The introduction of the coupling function and the reaction upon the elastic medium may bring important consequences in estimating the resonance regime of a structure (either natural or man-built) subjected to the action of a seismic motion. The present treatment opens the way of introducing various features in the coupling functions, in order to be in more realistic situations; in particular it is amenable to introducing the non-linearities which may affect the coupling of the structure with its site motion.

References

[1] P. M. Morse and H. Feschbach, *Methods of Theoretical Physics*, vol. 1, McGraw-Hill, NY (1953).

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