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Some relativistic notes

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Lorentz transformations. The electromagnetic wave equation $(1/c^2)\partial^2 f/\partial t^2 - \Delta f = 0$ in vacuum must preserve its form on changing the uniformly-moving reference frames (this is called the relativity principle). This amounts to keep invariant the interval $c^2t^2 - x^2$ for a frame moving with constant velocity v along the x-axis. Such an invariance is realized by a rotation of imaginary angle, which leads to the Lorentz transformations

$$x^{'} = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \ , \ t^{'} = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \ ; \tag{1}$$

here $x^{'}$and $t^{'}$ are the coordinate and, respectively, the time in the frame $K^{'}$ which moves with the constant velocity v (*i.e.* rectilinearly and uniformly) along the x-axis, with respect to the rest frame K, where we measure the coordinate x and the time t (Fig. 1).

We can see that the velocity of light c is a universal constant (with respect to uniformly-moving frames, which we call inertial); we can also see that it is the maximum velocity. For $v \ll c$ the Lorentz transformations become the Galileo transformations, $x^{'} = x - vt$, $t^{'} = t$. Sometimes, it is convenient to use the notations $\beta = v/c$ nd $\gamma = (1-\beta^2)^{-1/2}$. It is worth noting that the transverse coordinates y and z do not change.

Space contraction and time dilatation. We get from equations (1)

$$\Delta x^{'} = \frac{\Delta x}{\sqrt{1 - \beta^2}} \ ; \tag{2}$$

$\Delta x^{'}$ is the length at rest; it folows that a moving length Δx gets shorter. This is the space contraction. The time $t^{'}$ at the origin $x^{'} = 0$ ($x = vt$) is

$$t^{'} = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} = \frac{t - v^2t/c^2}{\sqrt{1 - v^2/c^2}} = \sqrt{1 - \beta^2}t \ ; \tag{3}$$

$t^{'}$is the time in the moving frame: it is shorter than in the rest frame. This is the time dilatation. $t^{'}$is also called the proper time.

Addition of velocities. We take the differentials of the equations (1) and get immediately

$$\begin{gathered} v_x^{'} = \frac{v_x - v}{1 - \frac{v}{c^2}v_x} \ , \\ v_y^{'} = \sqrt{1-\beta^2}\frac{v_y}{1 - \frac{v}{c^2}v_x} \ , \ v_z^{'} = \sqrt{1-\beta^2}\frac{v_z}{1 - \frac{v}{c^2}v_x} \ . \end{gathered} \tag{4}$$

We can see that the law of composition of the velocities is not commutative, any two velocities less than c combine to a velocity less than c, two parallel velocities c give one velocity c, etc.

Aberration of light. Let us assume that we have a source of light in the frame K, *i.e.* we assume

$$v_x^2 + v_y^2 + v_z^2 = v_0^2 = c^2 \; ; \tag{5}$$

we introduce the angles α and $\alpha^{'}$ through $v_x = v_0 \cos\alpha = c\cos\alpha$ and $v_x^{'} = v^{'}\cos\alpha^{'}$, where $v'^2 = v_x'^2 + v_y'^2 + v_z'^2$. We have

$$\begin{gathered} v_x^{'} = c\tfrac{\cos\alpha-\beta}{1-\beta\cos\alpha} \; , \\ v_y^{'} = \sqrt{1-\beta^2}\tfrac{v_y}{1-\beta\cos\alpha} \; , \; v_z^{'} = \sqrt{1-\beta^2}\tfrac{v_z}{1-\beta\cos\alpha} \end{gathered} \tag{6}$$

and

$$v^{'} = c\frac{\left[(\cos\alpha-\beta)^2 + (1-\beta^2)\sin^2\alpha\right]^{1/2}}{1-\beta\cos\alpha} = c \tag{7}$$

as expected. In addition,

$$v^{'}\cos\alpha^{'} = c\cos\alpha^{'} = v_x^{'} = c\frac{\cos\alpha-\beta}{1-\beta\cos\alpha} \; , \tag{8}$$

or

$$\cos\alpha^{'} = \frac{\cos\alpha-\beta}{1-\beta\cos\alpha} \; , \; \cos\alpha = \frac{\cos\alpha^{'}+\beta}{1+\beta\cos\alpha^{'}} \tag{9}$$

and

$$\sin\alpha^{'} = \sqrt{1-\beta^2}\frac{\sin\alpha}{1-\beta\cos\alpha} \; . \tag{10}$$

This is the aberration of light. Equations (9) and (10) give the deviation of a ray of light on passing from one frame to another. The phenomenon is shown in Figs. 2 and 3. On moving near the source $\cos\alpha < 0$ and $\cos\alpha^{'} < 0$ and $\cos\alpha^{'} < \cos\alpha$, according to equation (9). In this region the observer in the frame $K^{'}$sees the source deviated to the forward direction. This is called the "forward beaming". On moving away from the source $\cos\alpha > 0$, but $\cos\alpha^{'}$ can be positive or negative; for $\cos\alpha > \beta$ the ray of light is deviated backward, as shown in Fig. 3. For $\cos\alpha < \beta$ there is a shadow cone where the ligh does not reach the observer any longer. This can be seen easily for $\alpha = -\pi/2$.

We can see that on moving near the source we see an enhanced luminosity, as a result of the decrease of the solid angle inthe moving frame. Indeed, the solid angles are given by

$$\frac{d\Omega^{'}}{d\Omega} = \frac{d(\cos\alpha^{'})}{d(\cos\alpha)} = \frac{1-\beta^2}{(1-\beta\cos\alpha)^2} \; . \tag{11}$$

For a longitudinal beaming $\cos\alpha = -1$, we get $d\Omega^{'}/d\Omega = (1-\beta)/(1+\beta)$, which, in the (ultra-) relativistic limit, becomes $d\Omega^{'}/d\Omega \simeq 1/4\gamma^2$.

Doppler effect. The phase $\omega t - \mathbf{kr}$ must be invariant on changing the frame; we have

$$\omega^{'}t^{'} - \frac{1}{c}(x^{'}\cos\alpha^{'} + y^{'}\sin\alpha^{'}) = \omega t - \frac{1}{c}(x\cos\alpha + y\sin\alpha) \; ;$$

making use of the Lorentz transformations (1) we get

$$\omega^{'} = \omega\frac{1-\beta\cos\alpha}{\sqrt{1-\beta^2}} \tag{12}$$

and equations (9) and (10) of the aberration of light. Equation (12) is the Doppler effect: the frequency decreases on moving away from the source and increases on coming near the source. In addition, there is a transverse Doppler effect for $\cos\alpha = 0$.

Fresnel drag. Let light be propagated in a medium, which moves with velocity v in the same direction as light. The velocity of light in the medium is c/n, where n is the refraction index of the medium (Fig. 4). We are interested in the velocity $v_x = V$ of light in the laboratory frame. Equation (4) gives

$$V = \frac{c/n + v}{1 + v/cn} \simeq \frac{c}{n} + v(1 - 1/n^2) \ . \tag{13}$$

The refraction index corresponds to the wavelength in the moving medium; from equation (12) we have

$$\omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}} \simeq \omega(1-\beta) \tag{14}$$

and the wavelengths

$$\lambda' \simeq \lambda(1 + vn/c) \ ; \tag{15}$$

we have

$$\frac{c}{n(\lambda')} = \frac{c}{n} - \frac{c}{n^2}\frac{dn}{d\lambda}\frac{\lambda vn}{c} = \frac{c}{n} - \frac{\lambda v}{n}\frac{dn}{d\lambda} \ ; \tag{16}$$

introducing this correction in equation (13) we get

$$V \simeq \frac{c}{n} + v\left(1 - 1/n^2 - \frac{\lambda}{n}\frac{dn}{d\lambda}\right) \ ; \tag{17}$$

a moving medium drags the light along.

Spots, beaming and focusing. Let S be a source of light in motion, as shown in Fig. 5. For short distances it emits a parallel (paraxial) beam, which does not exhibit aberration. For longer distances, the dispersion gives a spot $\Delta S'$, which should be compared with the spot ΔS, corresponding to the source at rest. From equation (10) we have

$$\sin\alpha' = \sqrt{1-\beta^2}\frac{\sin\alpha}{1+\beta\cos\alpha} \tag{18}$$

and

$$\sin\alpha' = \sqrt{1-\beta^2}\frac{\sin\alpha}{1+\beta} = \sqrt{\frac{1-\beta}{1+\beta}}\sin\alpha \tag{19}$$

for small values of α and α' (the sign of the velocity is reversed in comparison with equations (9) and (10)). The ratio of the two spots is given by

$$\frac{\Delta S'}{\Delta S} = \frac{\alpha'^2}{\alpha^2} = \frac{1-\beta}{1+\beta} \ ; \tag{20}$$

the light beams in the forward direction; in the (ultra-) relativistic limit $\Delta S'/\Delta S \simeq 1/2\gamma^2$. Therefore, the flux of a laser light (energy per cross-sectional area) increases by a factor $2\gamma^2$, as a result of the light beaming. At the same time, the pulse duration of the laser decreases by a factor γ, so the laser intensity (flow of energy, *i.e.* energy per unit cross-sectional area and per unit time) increases by a factor $2\gamma^3$.

A pulsed, focused laser beam can be viewed as a succession of "slices" of light flying through the gometrical-optics frame of focalized light, the thickness of the slices being dictated by the pulse duration (Fig. 6). Visualized on a screen moving toward the focus F, the spot appears of shorter transverse dimensions, due to the aberration of light. According to the laws of the geometrical optics, we deduce that the transverse dimensions of the focus F are shorter, in keeping with the paradigm that we cannot focalize light over distances shorter than the wavelength (the wavelength gets shorter on moving near the focus). Similarly, on moving away from the focus, the wavelength gets longer and the spot gets wider, in accordance with a wider focus; again, we cannot focalize light on distances shorter than the wavelength.

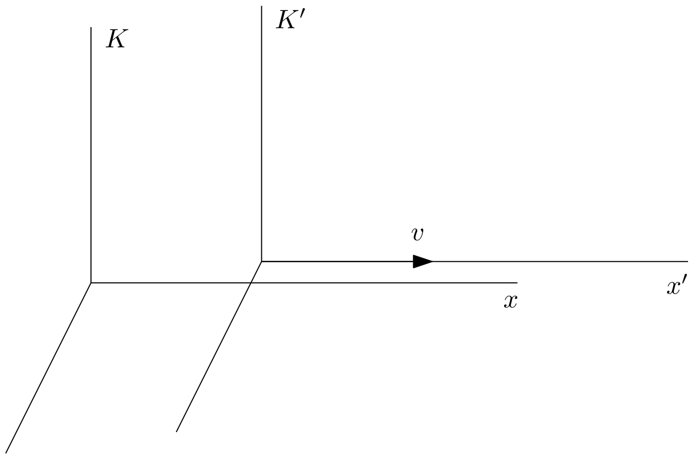
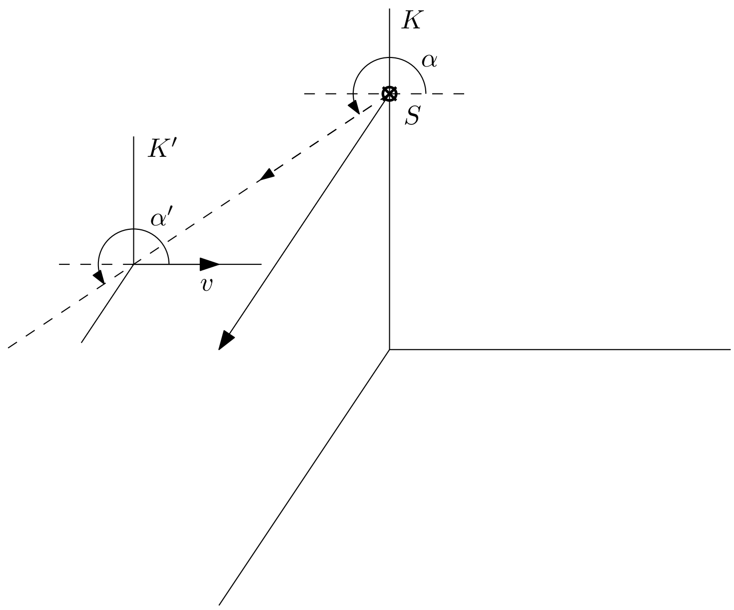


Fig. 1 Moving frames



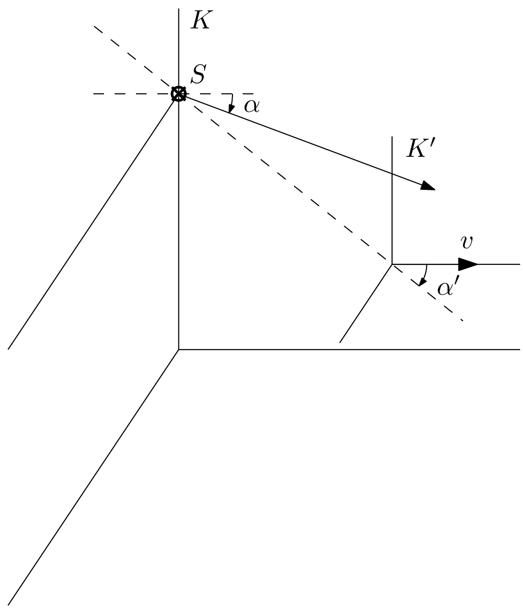


Fig. 2 Aberartion of light

Fig. 3 Aberration of light

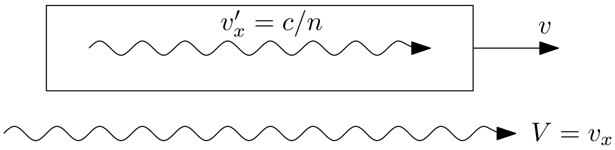


Fig. 4 Fresnel drag

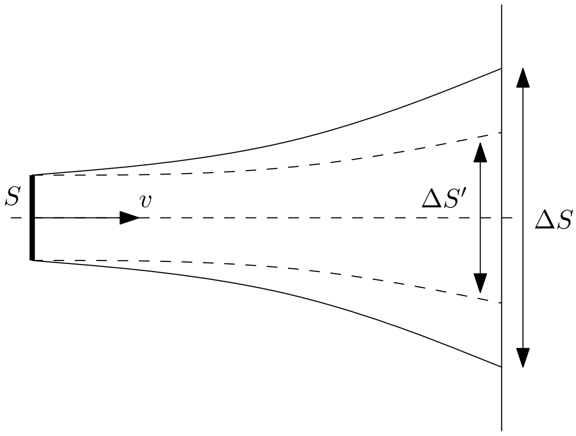


Fig. 5 Collimation - forward beaming

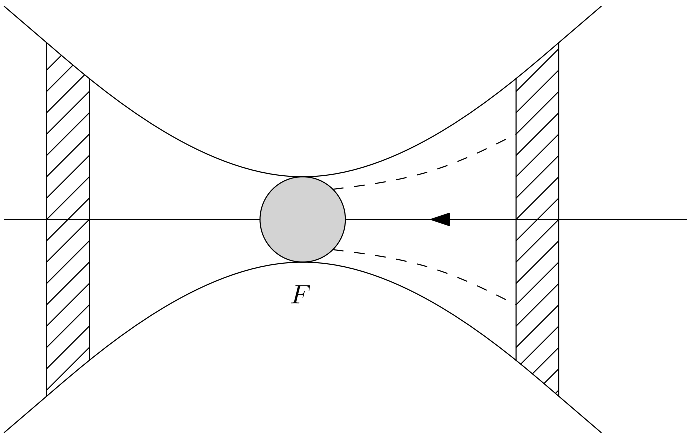


Fig. 6 Focusing - forward beaming