Journal of Theoretical Physics

Founded and Edited by M. Apostol **221** (2013)

ISSN 1453-4428

On the frequency shift of piezoelectric sensors and associated damping of the signal

B. F. Apostol
Department of Seismology, Institute of Earth's Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: afelix@theory.nipne.ro

The shift. We consider a homogeneous piezoelectric plate with thickness d and density ρ, with one fixed surface and another surface free. The wave equation which governs the waves propagating along the plate thickness is

$$\rho\frac{\partial^2 u}{\partial t^2}-K\frac{\partial^2 u}{\partial^2 x}=0\ , \tag{1}$$

where u is the displacement field, K is the elastic constant and $v=\sqrt{K/\rho}$ is the wave velocity. In typical solids the elastic wave velocity is $v\simeq 2000m/s$; for $1MHz$ a frequency we get a wavelength $\lambda\simeq 2mm$. The general solution of equation (1) can be represented as $u=Ae^{-i\omega t+ikx}+Be^{-i\omega t-ikx}$, where $\omega^2=v^2k^2$ and $A=B^*$, or $u=u_0\cos\omega t\cos(kx+\varphi)$. Leaving aside the temporal factor equation (1) becomes

$$\frac{\partial^2 u}{\partial x^2}+\frac{\rho}{K}\omega^2 u=0\ ; \tag{2}$$

imposing the boundary conditions $u(x=0)=0$ (fixed end) and $du/dx|_{x=d}=0$ (free end), we get

$$u_n=u_0\sin k_n x\ ,\ \ k_n d=(n+1/2)\pi\ ,\ \ n=0,\pm1,\pm2,...\ . \tag{3}$$

We assume now that a thin film of density $\delta\rho$ and thickness $h\ll d$ is deposited at the free surface; this amounts to a change $\rho\rightarrow\rho+\delta\rho h\delta(x-d)$ in the density; equation (2) becomes

$$\frac{\partial^2 u}{\partial x^2}+[\frac{\rho}{K}+\frac{\delta\rho}{K}h\delta(x-d)]\omega^2 u=0\ ; \tag{4}$$

the boundary condition at the free end is obtained now by integrating once equation (4); it reads

$$\left.\frac{\partial u}{\partial x}\right|_{x=d^+}-\left.\frac{\partial u}{\partial x}\right|_{x=d^-}+\frac{\delta\rho}{K}h\omega^2 u(x=d)=0\ ; \tag{5}$$

since u and its derivatives are vanishing for $x>d$, we get

$$\left.\frac{\partial u}{\partial x}\right|_{x=d^-}-\frac{\delta\rho}{K}h\omega^2 u(x=d)=0 \tag{6}$$

(which is a generalized Dirichlet-von Neumann boundary condition).

For $x<d$, the wave equation is not modified, so we have the solution $u=u_0\sin kx$ (and $\omega^2=v^2k^2$). The boundary condition (6) gives

$$\cot kd=\frac{\delta\rho}{\rho}\frac{h}{d}\cdot kd\ . \tag{7}$$

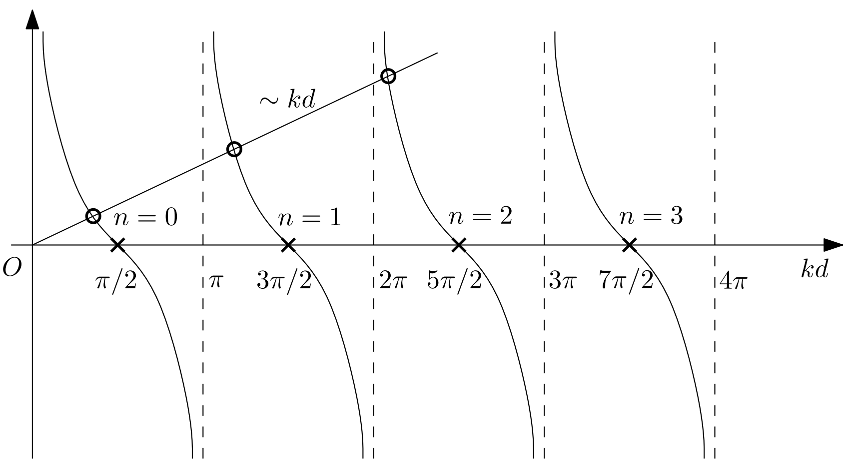

Figure 1: $\cot kd$ *vs* kd. The crossed points indicate the frequency without added mass, the encircled points correpond to the frequency shift.

This equation has a graphical solution, as shown in Fig. 1. We can see that the original frequencies $\omega_n = vk_n$, $k_n d = (n+1/2)\pi$ (equation (3), crossed points in Fig. 1) are shifted to the encircled points in Fig. 1. For $(\delta\rho/\rho)(h/d) > 0$ the frequency shift is negative, for $(\delta\rho/\rho)(h/d) < 0$ the frequency shift is positive. For $|(\delta\rho/\rho)(h/d)| \ll 1$ we can solve the equation (7) by means of the perturbation theory; indeed, we set

$$kd = k_n d + \delta_n \tag{8}$$

and get

$$\tan\delta_n \simeq \delta_n = -\frac{\delta\rho}{\rho}\frac{h}{d}(n+1/2)\pi \tag{9}$$

(for a limited range of n). For the fundamental mode $n = 0$ we get $\delta_0 = -\pi h\delta\rho/2\rho d$ and $\delta\omega_0/\omega_0 = -h\delta\rho/\rho d$; in general, $\delta\omega_n/\omega_n = -h\delta\rho/\rho d$ (as long as $|\delta\omega_n/\omega_n| \ll 1$).

We can increase the sensitivity by increasing, within reasonable limits, the parameter $|h\delta\rho/\rho d|$. In general, inhomogeneities in the plate or the deposited film produce noise, especially for their mean separation distance of the order of the wavelength. Another important cause of noise is the limited extension of the sensor along the transverse directions and its interfaces (contacts). All these mechanisms of wave damping are active for waves parallel with the sensor surfaces. In particular, the excitation of the surface waves (Rayleigh waves) can increase the damping, because these waves are have short wavelengths, which may be comparable with the mean separation length of the inhomogeneities dispersed in the surface film.

Damping. Let us consider a plane wave $u = u_0 e^{-i\omega t + i\mathbf{kr}}$ with usual notations and $\omega = ck$; leaving aside the temporal factor it satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0 \ . \tag{10}$$

On a scatterer localized at $\mathbf{r}_0$ the wave is

$$\delta u = u_0 v \delta(\mathbf{r} - \mathbf{r}_0) e^{i\mathbf{kr}_0} \ \ , \tag{11}$$

where v is the scatterer's volume. This localized wave is a source for the scattered waves. Inded, it acts with a "force"

$$(\Delta + \overline{k}^2)\delta u \tag{12}$$

upon the medium, where $\overline{k} = \omega/\overline{c}$ takes into account the nature of the scatterer through the modified velocity $\overline{c}$. The wave equation for the scattered waves is

$$\Delta u_s + k^2 u_s = (\Delta + \overline{k}^2)\delta u = (\Delta + \overline{k}^2) u_0 v \delta(\mathbf{r} - \mathbf{r}_0) e^{i\mathbf{k}\mathbf{r}_0} \ . \tag{13}$$

The solution of this equation is given by

$$u_s = \frac{1}{4\pi} \int d\mathbf{r}' [(\Delta' + \overline{k}^2) u_0 v \delta(\mathbf{r}' - \mathbf{r}_0) e^{i\mathbf{k}\mathbf{r}_0}] \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \ , \tag{14}$$

where the spherical wave $e^{ik|\mathbf{r}-\mathbf{r}'|}/\,|\mathbf{r}-\mathbf{r}'|$ is the Green function of the Helmholtz equation (in three dimensions). The Δ'-term can be integrated by parts, leading to the localized wave as expected. We are left with the scattered wave

$$u_s = \frac{u_0 v \overline{k}^2}{4\pi} e^{i\mathbf{k}\mathbf{r}_0} \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} \ . \tag{15}$$

For an assembly of scatteres located at $\mathbf{r}_i$ we get

$$u_s = \frac{u_0 v \overline{k}^2}{4\pi} \sum_i e^{i\mathbf{k}\mathbf{r}_i} \frac{e^{ik|\mathbf{r}-\mathbf{r}_i|}}{|\mathbf{r}-\mathbf{r}_i|} \ , \tag{16}$$

or

$$u_s = \frac{u_0 n v \overline{k}^2}{4\pi} \int d\mathbf{r}_i e^{i\mathbf{k}\mathbf{r}_i} \frac{e^{ik|\mathbf{r}-\mathbf{r}_i|}}{|\mathbf{r}-\mathbf{r}_i|} \ , \tag{17}$$

where n is the density of scatterers.

Let us assume that the original wave propagates along the x-direction, and let us set $\mathbf{r}_i = (x, \mathbf{R}_i)$; we are interested in the wave scattered by a slice of thickness ΔX, between X and $X + \Delta X$; we get

$$\Delta u_s = \frac{u_0 n v \overline{k}^2}{4\pi} \int d\mathbf{R}_i \int_{\Delta X} dx e^{ikx} \frac{e^{ik\sqrt{(\mathbf{R}-\mathbf{R}_i)^2 + (X-x)^2}}}{\sqrt{(\mathbf{R}-\mathbf{R}_i)^2 + (X-x)^2}} \ , \tag{18}$$

which can be approximated by

$$\Delta u_s = \frac{u_0 n v \overline{k}^2}{4\pi} \Delta X \int d\mathbf{R} \frac{e^{ikR}}{R} e^{ikX} = i \frac{u_0 n v \overline{k}^2}{2k} \Delta X e^{ikXx} = i \frac{n v \overline{k}^2}{2k} u(X) \Delta X \ . \tag{19}$$

Up to a phase factor (i), this is the loss of the incident wave, *i.e.*

$$\Delta u = -\frac{n v \overline{k}^2}{2k} u \Delta x \ , \tag{20}$$

which shows that the wave acquires a damping factor $u \rightarrow u^{-\alpha x}$, where

$$\alpha = \frac{1}{2} n v k \tag{21}$$

(for the present purpose we may take $\overline{k} = k$).

A similar estimation can be done in two dimensions, where it is convenient to represent the Green function $G(\mathbf{r})$ of the Helmholtz equation

$$\Delta G + k^2 G = \delta(\mathbf{r}) \tag{22}$$

as

$$G(\mathbf{r}) = \frac{1}{(2\pi)^2} \int d\mathbf{q} \frac{e^{i\mathbf{qr}}}{k^2 - q^2} \tag{23}$$

(Hankel function). The calculations are similar with those given above and the result is

$$\alpha = \frac{1}{2} n_s s k \ , \tag{24}$$

where n_s is the superficial density of the scatterers and s is their area.

From equations (21) and (24) we can see that the damping can be reduced by reducing the effective volume (area) nv ($n_s s$) of the scattering centres, and by using waves with long wavelengths, as expected.