Journal of Theoretical Physics

Founded and Edited by M. Apostol **224** (2013)

ISSN 1453-4428

QCD

M. Apostol
Department of Theoretical Physics,
Institute of Atomic Physics, Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

The Dirac equation

$$[\gamma^\mu(p_\mu-\frac{q}{c}A_\mu)-mc]\psi=0\ ,\ [i\gamma^\mu(\hbar\partial_\mu+i\frac{q}{c}A_\mu)-mc]\psi=0 \tag{1}$$

for a particle of charge q and mass m in the external electromagnetic field with potential A_μ is generalized to

$$[i\gamma^\mu(\hbar\partial_\mu-ig\frac{q}{c}t_aA^a_\mu)-m_fc]\psi_f=0 \tag{2}$$

in Quantym Chromodynamics (QCD), where t_a, $a=1,2...8$ are the generators of the $SU(3)$ group,

$$[t_a,t_b]=if^c_{ab}t_c\ \ , \tag{3}$$

and f (up, down, strange, charm, top, bottom) are the flavour labels of the quark fields ψ_f; A^a_μ are the potentials of the gluon field, t_a are 3×3-matrices (color labels), ψ_f are color bispinor and f^c_{ab} are the structure factors of the $SU(3)$ group; γ^μ are the Dirac matrices. Equation (2) is generated by the particle (quark) lagrangian which, with usual conventions, can be written as

$$L_q=\overline{\psi}_f[i\gamma^\mu(c\hbar\partial_\mu-igqt_aA^a_\mu)-m_fc^2]\psi_f\ . \tag{4}$$

We note the change of sign of the g-term with respect to the Dirac equation, as well as the presence of the charge factor q in the g-term, where $-q$ is set equal with the electron charge $-e$. The gluon lagrangian reads

$$L_g=-\frac{1}{16\pi}G^a_{\mu\nu}G^{\mu\nu}_a\ ,\ G^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^a_{bc}A^b_\mu A^c_\nu\ \ , \tag{5}$$

where $G^a_{\mu\nu}$ are the gluon fields. The variation of the action with respect to the potentials A^a_μ leads to the field equations

$$\partial_\nu G^{\mu\nu}_a+gf^b_{ac}G^{\mu\nu}_bA^c_\nu=4\pi gq\overline{\psi}_f\gamma^\mu t_a\psi_f\ ; \tag{6}$$

$-cgq\overline{\psi}_f\gamma^\mu t_a\psi_f$ can be viewed as a density of current j^μ_a.

Since $\partial_\mu=(\partial/c\partial t,\partial_i)$ equation (2) can also be writen as

$$i\hbar\frac{\partial\psi}{\partial t}=[-ic\hbar\gamma^0\gamma^i\partial_i-gq\gamma^0\gamma^\mu tA_\mu+mc^2\gamma^0]\psi\ \ , \tag{7}$$

where summation over a is included and the flavor label is left aside. Now we use

$$\gamma^0=\beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\ ,\ \gamma^0\gamma^i=\alpha^i=\begin{pmatrix}0&\sigma^i\\\sigma^i&0\end{pmatrix}\ \ , \tag{8}$$

where σ^i are the Pauli matrices; and we use also $-ic\hbar\gamma^i\partial_i = -c\gamma^i p_i = c\gamma^i p^i = c\overrightarrow{\alpha}\mathbf{p}$, where $\mathbf{p} = -i\hbar grad$ is the momentum; we get

$$i\hbar\frac{\partial\psi}{\partial t} = [c\overrightarrow{\alpha}\mathbf{p} - gqt\Phi + gqt\overrightarrow{\alpha}\mathbf{A} + mc^2\beta]\psi \ , \tag{9}$$

where $A_\mu = (\Phi, -\mathbf{A})$; here we set $\psi \sim e^{-i(mc^2/\hbar)t}$ and write $\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$,

$$\begin{aligned} i\hbar\tfrac{\partial\varphi}{\partial t} + mc^2\varphi &= c\overrightarrow{\sigma}(\mathbf{p} + g\tfrac{q}{c}t\mathbf{A})\chi - gqt\Phi\varphi + mc^2\varphi \ , \\ i\hbar\tfrac{\partial\chi}{\partial t} + mc^2\chi &= c\overrightarrow{\sigma}(\mathbf{p} + g\tfrac{q}{c}t\mathbf{A})\varphi - gqt\Phi\chi - mc^2\chi \ . \end{aligned} \tag{10}$$

In the non-relativistic limit we get

$$\begin{aligned} \chi &\simeq \tfrac{1}{2mc}\overrightarrow{\sigma}(\mathbf{p} + g\tfrac{q}{c}t\mathbf{A})\varphi \ , \\ i\hbar\tfrac{\partial\varphi}{\partial t} + gqt\Phi\varphi &= \tfrac{1}{2m}[\overrightarrow{\sigma}(\mathbf{p} + g\tfrac{q}{c}t\mathbf{A})]^2\varphi \ ; \end{aligned} \tag{11}$$

here we use the identity $\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$ and get finally

$$\begin{aligned} i\hbar\tfrac{\partial\varphi}{\partial t} + gqt_a\Phi^a\varphi &= \tfrac{1}{2m}[(\mathbf{p} + g\tfrac{q}{c}t_a\mathbf{A}^a)]^2\varphi + \\ +g\tfrac{\hbar q}{2mc}\overrightarrow{\sigma}t_a curl\mathbf{A}^a\varphi &- g^2\tfrac{q^2}{4mc^2}f^c_{ab}t_c\overrightarrow{\sigma}(\mathbf{A}^a \times \mathbf{A}^b)\varphi \ . \end{aligned} \tag{12}$$

This is the analogue of the Pauli equation for strong interactions, where we can see a "magnetic" field $curl\mathbf{A}^a$, another "magnetic" field $\sim \mathbf{A}^a \times \mathbf{A}^b$ and the quark spin $\sim \overrightarrow{\sigma}$. The last term in equation (12) can be omitted within the present approximation, since it is of second order in powers of $1/c$.

We define the current density

$$j^\mu_a = cq\overline{\psi}\gamma^\mu t_a\psi \tag{13}$$

and get the charge density

$$\rho_a = j^0_a \simeq cq\varphi^* t_a\varphi \tag{14}$$

in the non-relativistic limit. For the current density

$$\mathbf{j}_a = cq\psi^*\overrightarrow{\alpha}t_a\psi = cq(\varphi^* t_a\overrightarrow{\sigma}\chi + \chi^* t_a\overrightarrow{\sigma}\varphi) \tag{15}$$

in the non-relativistic limit we use χ given by equation (11) and the identity given above for $\sigma_i\sigma_j$; we get

$$\begin{aligned} \mathbf{j}_a &\simeq \tfrac{q}{2m}[\varphi^* t_a(\mathbf{p}\varphi) - (\mathbf{p}\varphi^*)t_a\varphi] + \\ &+\tfrac{gq^2}{2mc}[\varphi^* t_a(t_b\mathbf{A}^b)\varphi + \varphi^*(t_b\mathbf{A}^b)t_a\varphi] + \tfrac{q\hbar}{2m}curl(\varphi^* t_a\overrightarrow{\sigma}\varphi) \ . \end{aligned} \tag{16}$$

Equation (6) becomes

$$\partial_\nu G^{\mu\nu}_a + gf^b_{ac}G^{\mu\nu}_b A^c_\nu = \frac{4\pi}{c}gj^\mu_a \tag{17}$$

with summation over all the flavours in the current density; or

$$\Box A^\mu_a - \partial^\mu\partial_\nu A^\nu_a - gf^{bc}_a\partial_\nu(A^\mu_b A^\nu_c) - gf^b_{ac}G^{\mu\nu}_b A^c_\nu = -\frac{4\pi}{c}gj^\mu_a \ . \tag{18}$$

We are interested in the non-relativistic limit of this equation ($c\rightarrow\infty$). We get

$$-\partial_i\partial^i\Phi_a+gf_a^{bc}\Phi_b\partial_iA_a^i+g^2(A^2)_a^d\Phi_d=4\pi gq\varphi_f^*t_a\varphi_f \ ,$$

$$-\partial_j\partial^jA_a^i+\partial^i\partial_jA_a^j+gf_a^{bc}\partial_j(A_b^iA_c^j)+ \tag{19}$$

$$+gf_{ac}^b(\Phi_c\partial^i\Phi_b+A_j^c\partial^iA_b^j-A_j^c\partial^jA_b^i)+g^2[(A^2)_a^d-(\Phi^2)_a^d]A_d^i=0 \ ,$$

where

$$(\Phi^2)_a^d=f_e^{db}f_{ac}^e\Phi_b\Phi^c \ , \ (A^2)_a^d=f_e^{db}f_{ac}^eA_b^jA_j^c \ . \tag{20}$$

In the limit $g\gg1$ we should have $(A^2)_a^d-(\Phi^2)_a^d=0$.

The condition $(A^2)_a^d-(\Phi^2)_a^d=0$ can also be written as $F_{a\mu}^bF_b^{d\mu}=0$, where $F_{a\mu}^b=f_{ac}^bA_\mu^c$; making use of the explicit structure factors f_{ac}^b ($123=1$, $147=246=257=345=1/2$, $156=367=-1/2$, $458=678=\sqrt{3}/2$) and computing the matrices F and F^2 we can see that we should have $A_\mu^a=0$, $a=1,2...7$. This makes 28 conditions for 32 unknowns. Consequently, we keep only one label a and rewrite the whole theory as

$$[i\gamma^\mu(\hbar\partial_\mu-ig\frac{q}{c}tA_\mu)-m_fc]\psi_f=0 \ , \tag{21}$$

$$L_q=\overline{\psi}_f[i\gamma^\mu(c\hbar\partial_\mu-igqtA_\mu)-m_fc^2]\psi_f \ , \tag{22}$$

$$L_g=-\frac{1}{16\pi}G_{\mu\nu}G^{\mu\nu} \ , \ G_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu \ , \tag{23}$$

$$\partial_\nu G^{\mu\nu}=4\pi gq\overline{\psi}_f\gamma^\mu t\psi_f \ . \tag{24}$$

This is electromagnetism with color charges ($\overline{\psi}t\psi$, and flavour charges), which have no effect at this level. We have, in effect, $G^{\mu\nu}=(\mathbf{E},\mathbf{H})$ and

$$div\mathbf{E}=4\pi\rho \ , \ div\mathbf{H}=0 \ ,$$

$$curl\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} \ , \ curl\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi}{c}\mathbf{j} \ , \tag{25}$$

$$\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}-grad\Phi \ , \ \mathbf{H}=curl\mathbf{A} \tag{26}$$

and

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi-\Delta\Phi=4\pi\rho \ ,$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}-\Delta\mathbf{A}=\frac{4\pi}{c}\mathbf{j} \ , \tag{27}$$

where we have imposed the Lorenz gauge. In the non-relativistic limit we may take $\mathbf{A}=0$ and

$$\Delta\Phi=4\pi gq\varphi_f^*t\varphi_f \ ,$$

$$i\hbar\frac{\partial\varphi_f}{\partial t}=-\frac{\hbar^2}{2m_f}\Delta\varphi_f-gqt\Phi\varphi_f \ . \tag{28}$$

It is easy to see that the system governed by these equations is unstable. Indeed,

$$i\hbar\int d\mathbf{r}\varphi_f^*\frac{\partial\varphi_f}{\partial t}=-\frac{\hbar^2}{2m_f}\int d\mathbf{r}\varphi_f^*\Delta\varphi_f-\frac{1}{4\pi}\int d\mathbf{r}\Phi\Delta\Phi>0 \ , \tag{29}$$

as expected from a self-interaction.