Journal of Theoretical Physics

Founded and Edited by M. Apostol **225** (2013)

ISSN 1453-4428

On the damping coefficient of the oscillators and waves

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Abstract

The damping coefficients of harmonic oscillators or waves (attenuation coefficient) may depend on the amplitude and frequency as a consequence of including non-linearities. For higher frequencies these coefficients may vanish, leading to disruption of the oscillations, or appreciable enhancement of the resonance.

As it is well known the equation of motion of a linear harmonic oscillator

$$\ddot{u}+\omega_0^2 u=0 \tag{1}$$

has the general solution $u=Ae^{i\omega_0 t}$ (A complex in general) or $u=A\cos(\omega_0 t+\varphi)$, where u is the motion coordinate, ω_0 is the eigenfrequency, A is the amplitude and φ is an initial phase; the constants A and φ are determined from the initial conditions (coordinate u and velocity $\dot{u}$ at the initial moment). A more complex oscillator may include a damping coefficient γ,

$$\ddot{u}+\omega_0^2 u+\gamma\dot{u}=0 \ , \tag{2}$$

as well as non-linear contributions. The solution of equation (2) is given by

$$u=Ae^{-\gamma t/2}\cos(\omega t+\varphi) \ , \ \omega=\sqrt{\omega_0^2-\gamma^2/4} \ ; \tag{3}$$

we assume $\omega_0>\gamma/2$. As it is well known, non-linear terms in powers of u in equation (2) lead to oscillations with combined frequencies, frequency shifts or resonances at rational multiples of ω_0. Here we investigate the effect of the presence of the non-linearities in the damping coefficient γ. For small amplitudes, we may assume a general non-linear damping coefficient of the form

$$\gamma=\gamma_0+\gamma_1\overline{u^2}+\gamma_2\overline{\dot{u}^2}+... \ , \tag{4}$$

where the averages in equation (4) are taken over the non-damped solution. We get

$$\overline{u^2}=\frac{1}{T}\int_{-T/2}^{T/2}dtA^2\cos^2(\omega_0 t+\varphi)=\frac{1}{2}A^2 \tag{5}$$

and

$$\overline{\dot{u}^2}=\frac{1}{T}\int_{-T/2}^{T/2}dtA^2\omega_0^2\sin^2(\omega_0 t+\varphi)=\frac{1}{2}A^2\omega_0^2 \ , \tag{6}$$

where T is a very long interval of time. The damping coefficient becomes

$$\gamma=\gamma_0+\frac{1}{2}\gamma_1 A^2+\frac{1}{2}\gamma_2 A^2\omega_0^2+... ; \tag{7}$$

we can see that it acquires a frequency dependence. Depending on the sign of the coefficients $\gamma_{0,1,2}$, the damping coefficient γ may increase for high frequencies, and the oscillations get more damped, or, it may decrease, vanish, and acquire negative values, and the oscillations are disrupted.

An oscillator acted by a periodic external force $f\cos\omega t$,

$$\ddot{u}+\omega_0^2 u+\gamma\dot{u}=f\cos\omega t \ , \tag{8}$$

has the particular solution

$$u=-\frac{1}{2}\frac{f}{\omega^2-\omega_0^2+i\omega\gamma}e^{-i\omega t}+c.c. \ ; \tag{9}$$

at resonance ($\omega=\omega_0$)

$$u=\frac{if}{2\omega_0\gamma}e^{-i\omega_0 t}+c.c. \ , \tag{10}$$

and we can see that for a vanishing γ the forced oscillations can be enhanced appreciably.

Let us consider the wave equation in one dimension

$$\ddot{u}-v^2u^{''}=0 \ , \tag{11}$$

where v is the wave velocity. The solution of equation (11) is the wave $u=A\cos(\omega t-kx+\varphi)$ where $\omega=vk$, or $u=A\cos\omega(t-x/v+t_0)$; this is a wave propagating along the x-direction, with an initial phase φ, related to the initial time t_0. The damping of the wave is included in the equation

$$\ddot{u}-v^2u^{''}+\gamma\dot{u}-\beta u^{'}=0 \ , \tag{12}$$

where

$$\begin{aligned}\gamma&=\gamma_0+\gamma_1\overline{u^2}+\gamma_2\overline{\dot{u}^2}+\gamma_3\overline{u'^2}+... \ ,\\ \beta&=\beta_0+\beta_1\overline{u^2}+\beta_2\overline{\dot{u}^2}+\beta_3\overline{u'^2}+... \ .\end{aligned} \tag{13}$$

We get the solution

$$u=A\cos(\omega t-kx+\varphi)e^{-\gamma t/2}e^{-\beta x/2v^2} \ , \ \omega=\sqrt{(\beta^2/4v^2-\gamma^2/4)+v^2k^2} \tag{14}$$

(with $\beta^2/v^2-\gamma^2>0$) and

$$\begin{aligned}\gamma&=\gamma_0+\tfrac{1}{2}\gamma_1A^2+\tfrac{1}{2}\gamma_2A^2\omega_0^2+\tfrac{1}{2}\gamma_3A^2\omega_0^2/v^2+... \ ,\\ \beta&=\beta_0+\tfrac{1}{2}\beta_1A^2+\tfrac{1}{2}\beta_2A^2\omega_0^2+\tfrac{1}{2}\beta_3A^2\omega_0^2/v^2+... \ .\end{aligned} \tag{15}$$

We can see that the damping coefficients depend on amplitude and frequency. Usually, we may take $\gamma=0$, and retain only the attenuation coefficient β. Depending on the sign and magnitude of the coefficients $\beta_{0,1,2,3}$ we can see that the wave can be disrupted for certain frequencies (wavenumbers).