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On the Bethe ansatz

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Abstract

It is shown that the Bethe ansatz is a wrong solution to a differential equation, violates the principle of particle identity, and infringes upon the wavefunction continuity.

In 1931 Bethe[1] attempted to solve the magnetic chains by what later became known as the Bethe ansatz. A reversed spin in such a chain is a wave, or a quantum particle, called magnon, and Bethe realized that the problem is a two-particle problem with a hamiltonian

$$H = t(x_1) + t(x_2) + U(x_1 - x_2) \quad , \tag{1}$$

where $t(x_{1,2})$ are kinetic terms and $U(x_1 - x_2)$ is a short-range interaction. Later on, the same problem has been recognized in one-dimensional ensembles of many particles interacting by δ-potentials, $U(x_1 - x_2) = U\delta\,(x_1 - x_2)$, and the Bethe ansatz has been applied to bosons,[2]′[3] and to attractive[4] and repulsive[5] fermions. The Hubbard model[6] in one dimension has also been approached by Bethe ansatz,[7] and the efforts of understanding the meaning of these computations are nowadays known as the "strongly-correlated electrons" problem.[8]′[9]

We show in this Note that the Bethe ansatz is a wrong "solution" to a differential equation, violates the principle of particle identity, and infringes upon the wavefunctions continuity.

Bethe[1] remarked that, since $U(x_1 - x_2) = 0$ for $x_1 < x_2$, the wavefunction of (1) can be taken as a product of plane waves,

$$\varphi(x_1, x_2) = \langle x_1 x_2 \mid k_1 k_2 \rangle \quad , \quad x_1 < x_2 \quad , \tag{2}$$

where $k_{1,2}$ are wavevectors; the corresponding "eigenvalue" of H is $t(k_1) + t(k_2)$. For $x_1 > x_2$ the interaction vanishes again, and, strangely, Bethe takes here the wavefunction as $A\varphi(x_1, x_2)$, where A remains to be determined. The Bethe ansatz begins therefore, wrongly, with

$$\psi(x_1, x_2) = \begin{array}{ll} \langle x_1 x_2 \mid k_1 k_2 \rangle \quad , & x_1 < x_2 \quad , \\ A\,\langle x_1 x_2 \mid k_1 k_2 \rangle \quad , & x_1 > x_2 \quad , \end{array} \tag{3}$$

for the motion of the particle pair. This is a wrong "solution" to the differential equation of the eigenvalue problem. The technical literature calls it the "diffractionless" of the Bethe ansatz.[10] The coefficient A is determined from the equation at $x_1 = x_2$, and ψ is symmetrized as for identical particles. The output is a discontinous ψ at $x_1 = x_2$, as if this point would be a boundary for the problem. Indeed, the technical literature calls it so.[11] (Sometimes.the steps are reversed, *i.e.* ψ is symmetrized first and A is determined thereafter; now ψ is continuous, but its derivatives are not at $x_1 = x_2$). The ansatz infringes, therefore, upon the wavefunctions continuity. Finaly, the principle of particle identity is violated, since

$$\psi(x_2, x_1) = \begin{array}{ll} A \langle x_2 x_1 \mid k_1 k_2 \rangle \ , & x_1 < x_2 \ , \\ \langle x_2 x_1 \mid k_1 k_2 \rangle \ , & x_1 > x_2 \ , \end{array} \tag{4}$$

(up to phase factor) is not a solution, if $\psi(x_1, x_2)$ is one. When the symmetrization is taken first, as, for instance,

$$\psi_s(x_{1,}x_2) = \psi(x_1, x_2) + \psi(x_2, x_1) =$$
$$= \begin{array}{ll} \langle x_1 x_2 \mid k_1 k_2 \rangle + A \langle x_2 x_1 \mid k_1 k_2 \rangle \ , & x_1 < x_2 \ , \\ A \langle x_1 x_2 \mid k_1 k_2 \rangle + \langle x_2 x_1 \mid k_1 k_2 \rangle \ , & x_1 > x_2 \ , \end{array} \tag{5}$$

the antisymmetrical ψ is not obtained for the same A; which shows again that the principle of particle identity is violated. The technical literature refers to this violation as to the "kaleidoscopy" of the Bethe ansatz.[10]

The correct procedure is the following. If $\varphi(x_1, x_2) = \langle x_1 x_2 \mid k_1 k_2 \rangle$ is a solution for $x_1 < x_2$ then $\chi(x_1, x_2) = \langle x_1 x_2 \mid k_3 k_4 \rangle$ is a solution for $x_1 > x_2$, where $t(k_3) + t(k_4) = t(k_1) + t(k_2)$ and $k_1 + k_2 = k_3 + k_4$ for continuity. The equation is thereafter solved at $x_1 = x_2$, and, finally, the solution is "symmetrized"; more exactly, the symmetry under the particle permutations is read on the solutions. Of course, the usual course is to start with the symmetrized free solution $\varphi(x_2, x_1) = \pm\chi(x_1, x_2) = \pm \langle x_1 x_2 \mid k_3 k_4 \rangle$, whence $k_3 = k_2$, $k_4 = k_1$, and solve the interacting problem; as it has been done for the Hubbard model in Ref.12.

The coefficient A in the Bethe ansatz is a phase shift, and much of the efforts in dealing with the strongly-correlated electrons "problem" are directed toward revealing the "meaning" of this phase shift. The question is "technically" related to the function

$$f(E) = -\frac{U}{N} \sum_k \frac{1}{t(k) + t)q - k) - E} \tag{6}$$

in the Hubbard model, where N is the number of lattice sites, which is not read as the eigenvalue equation $f(E) = 1$, but as a p.v. integral, whose "meaning" remains to be "established".[8]′[9] Other "technical" connections with the particle collisions are obvious in the Bethe ansatz.

One may say that, indeed, the electrons are "strongly correlated" by the Bethe ansatz, by requiring $k_3 = k_1$, $k_4 = k_2$, by introducing the boundary at $x_1 = x_2$, and by restricting the symmetry under particle permutations.

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