Journal of Theoretical Physics

Founded and Edited by M. Apostol **25** (1997)

ISSN 1453-4428

On the diffusion equation

M. Apostol
Department of Theoretical Physics,
Institute of Atomic Physics,
Magurele-Bucharest MG-6,
POBox MG-35, Romania
email: apoma@theor1.ifa.ro

Abstract

The meaning of the diffusion equation is stated and various forms of its solution are given.

Suppose that a number of particles are placed at the origin $x = 0$ of the space at the initial moment of the time $t = 0$, such that their density is $n(x, t = 0) = \delta(x)$; suppose that these particles diffuse to the right. Here, by diffusion is meant that the particles move with the probability λ in the time unit over the space unit. Under these circumstances we can write down the diffusion equation

$$n(x, t+1) - n(x, t) = \lambda \left[n(x-1, t) - n(x, t) \right] \quad , \tag{1}$$

where x and t stand for distance and, respectively, time, both measured in their own units. Since λ depends on the space and time units the diffusion process described by the above equation is inconsistent; the only way to give a meaning to this equation is to assume that the unit of the time is infinitesimal, *i.e.* $t \to \infty$ (or $t \gg 1$), in which case $\lambda \ll 1$. Remark that this is not possible for the space unit. Under this assumption $n(x, t)$ is a slow function, and, according to the above equation, it is also a smooth function; which means that $t \gg x \gg 1$.

Under these assumptions the above equation may be written as

$$\frac{\partial n}{\partial t} = \lambda \left[-\frac{\partial n}{\partial x} + \frac{1}{2}\frac{\partial^2 n}{\partial x^2} + ... \right] \quad . \tag{2}$$

Introducing the Fourier transform

$$n(x, t) = \frac{1}{2\pi} \int dq \cdot n(q, t) e^{iqx} \quad , \tag{3}$$

and restricting ourselves to the first term in the right-hand side of (2) we get $n(x, t) = \delta(x - \lambda t)$, *i.e.* a purely propagating distribution of particles, moving with the velocity λ. This function is not a smooth one near the propagating front, so that, in order to improve the accuracy, we keep also the second derivative in (2); in this case we obtain the well-known gaussian

$$n(x, t) = \frac{1}{(2\pi\lambda t)^{1/2}} e^{-\frac{(x-\lambda t)^2}{2\lambda t}} \quad , \tag{4}$$

which has a propagating front $x \sim \lambda t$ and a dispersion $\sim 2\sqrt{\lambda t}$. For $t \gg x \gg 1$ this is a slow and smooth function.

If we keep all the terms in the Taylor expansion in (2) we obtain

$$\frac{\partial}{\partial t}n(q,t)=\lambda\left(e^{-iq}-1\right)n(q,t)\quad , \tag{5}$$

whence

$$n(x,t)=\frac{1}{2\pi}\int dq\cdot e^{\Phi(q)}\quad , \tag{6}$$

where

$$\Phi(q)=iqx+\lambda t\left(e^{-iq}-1\right)\quad . \tag{7}$$

The integral in (6) is estimated by the method of the steepest descent, the main contribution to it coming, thereby, from the smoothest part of the function under integral. We have

$$\Phi(q)=\Phi\left(q_0\right)+\frac{1}{2}\left(q-q_0\right)^2\Phi''\left(q_0\right)\quad , \tag{8}$$

where $\Phi'\left(q_0\right)=0$,

$$q_0=i\ln\left(x/\lambda t\right)\quad ,\quad x\neq 0\quad . \tag{9}$$

We obtain

$$n(x,t)=\frac{1}{\sqrt{2\pi x}}e^{x-\lambda t-x\ln(x/\lambda t)}\quad , \tag{10}$$

a function which looks completely different from the gaussian given above. However, this function has a maximum $1/\sqrt{2\pi\lambda t}$ placed at $x=\lambda t$, and a dispersion $\Delta x=2\sqrt{\lambda t}$, *i.e.* it is but another representation of the gaussian. (The singularity exhibited by this function at the origin is spurious). More than this, its integral over the entire space is about $\sqrt{2/\pi}$, which is close to the unity, showing that the steepest descent method is fairly accurate.

Actually, equation (1) has an exact solution

$$n(x,t)=\frac{1}{x!}t\left(t-1\right)...\left(t+1-x\right)\lambda^x\left(1-\lambda\right)^{t-x}=C_t^x\lambda^x\left(1-\lambda\right)^{t-x}\quad , \tag{11}$$

for $t\geq x\geq 1$, $n(0,t)=\left(1-\lambda\right)^t$, and zero otherwise, which looks again completely different from the above two approximate solutions. However, apart from the fact that $\overline{x}=\lambda t$ and $\overline{\left(\Delta x\right)^2}\equiv\lambda t$, where the averages are taken with the binomial distribution given above, using the well-known approximate representations for the factorial

$$\begin{aligned} t! \quad &\sim\quad t^t\ ,\ t\gg 1\quad ,\\ \ln x! \quad &\sim\quad x\ln x-x+\frac{1}{2}\ln x+\ln\sqrt{2\pi}\ ,\ x\gg 1\quad , \end{aligned} \tag{12}$$

equation (11) becomes (10) for $t\gg x\gg 1$.