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On unphysical terms in the elastic Hertz potentials

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Abstract

Unphysical terms in the elastic Hertz potentials are identified and a regularization procedure is devised for removing them. The solutions of the equation of the elastic motion are given for tensorial forces (seismic moment forces) and vectorial forces (Stokes problem) concentrated both in space and time.

We consider the equation for the elastic motion[1]

$$\ddot{\mathbf{u}}-c_t^2\Delta\mathbf{u}-(c_l^2-c_t^2)grad\cdot div\mathbf{u}=\mathbf{F}\ \ , \tag{1}$$

where $\mathbf{u}$ is the displacement, $c_{l,t}$ are the velocities of the elastic waves (longitudinal and transverse) and $\mathbf{F}$ is the force (per unit mass); we consider a force concentrated (localized) both in space and time, given by[2]

$$F_i=m_{ij}T\delta(t)\partial_j\delta(\mathbf{R})\ \ , \tag{2}$$

where T is the short duration of the time impulse $\delta(t)$ and m_{ij} is the tensor of the seismic moment. We follow the standard procedure for introducing the Hertz vectors.[3] To this end we introduce the notation $f_i=-(1/4\pi)m_{ij}T\partial_j$ and write the equation as

$$\ddot{\mathbf{u}}-c_t^2\Delta\mathbf{u}-(c_l^2-c_t^2)grad\cdot div\mathbf{u}=-4\pi\delta(t)\mathbf{f}\delta(\mathbf{R})\ ; \tag{3}$$

then, we write $\delta(\mathbf{R})=-(1/4\pi)\Delta\frac{1}{R}$; the equation becomes

$$\ddot{\mathbf{u}}-c_t^2\Delta\mathbf{u}-(c_l^2-c_t^2)grad\cdot div\mathbf{u}=\delta(t)\Delta\frac{\mathbf{f}}{R}\ ; \tag{4}$$

further on, we use $\Delta(\mathbf{f}/R)=grad\cdot div(\mathbf{f}/R)-curl\cdot curl(\mathbf{f}/R)$ and get

$$\ddot{\mathbf{u}}-grad\cdot div[c_l^2\mathbf{u}+\delta(t)\frac{\mathbf{f}}{R}]+curl\cdot curl[c_t^2\mathbf{u}+\delta(t)\frac{\mathbf{f}}{R}]=0\ ; \tag{5}$$

we can see that $\mathbf{u}$ can be written as

$$\mathbf{u}=grad\cdot div\mathbf{B}+curl\cdot curl\mathbf{C}=-\Delta\mathbf{C}+grad\cdot div(\mathbf{B}+\mathbf{C})\ \ , \tag{6}$$

where

$$\ddot{\mathbf{B}}-c_l^2\Delta\mathbf{B}=\delta(t)\frac{\mathbf{f}}{R}\ ,\ \ddot{\mathbf{C}}-c_t^2\Delta\mathbf{C}=-\delta(t)\frac{\mathbf{f}}{R}\ ; \tag{7}$$

the vectors $\mathbf{B}$ and $\mathbf{C}$ are known in **Electromagnetism** as the Hertz vectors.[4]-[6]

We are led to study the potential equation

$$\ddot{\Psi} - c^2\Delta\mathbf{\Psi} = \delta(t)\frac{1}{R} \ , \tag{8}$$

where the vectors $\mathbf{B}$, $\mathbf{C}$ are given by $\mathbf{f}\Psi$. For the tensorial force given in equation (2) we get the potentials $\mathbf{B}$ and $\mathbf{C}$ by applying the operator $f_i = -(1/4\pi)m_{ij}T\partial_j$ to the scalar potential Ψ; the displacement is given by

$$u_i = -\frac{T}{4\pi}m_{ij}\partial_j\Delta\Psi_t - \frac{T}{4\pi}m_{jk}\partial_i\partial_j\partial_k(\Psi_l - \Psi_t) \ , \tag{9}$$

where $\Psi_{l,t}$ correspond to $c_{l,t}$ in equation (8). For a vectorial force $\mathbf{F} = \mathbf{f}T\delta(t)\delta(\mathbf{R})$ (Stokes problem[7]) we get the potentials $\mathbf{B}$ and $\mathbf{C}$ by applying the constant vector $-(1/4\pi)T\mathbf{f}$ to Ψ; from equation (6) the displacement is given by

$$\mathbf{u} = -\frac{T}{4\pi}\mathbf{f}\Delta\Psi_t - \frac{T}{4\pi}grad(\mathbf{f}grad)(\Psi_l - \Psi_t) \ . \tag{10}$$

The solution of equation (8) is of the form $\Psi = \chi(R,t)/R$, where

$$\ddot{\chi} - c^2\chi^{''} = \delta(t) \ ; \tag{11}$$

we get

$$\Psi = \frac{t\theta(t)}{R} + \theta(ct - R)\frac{\chi(R - ct)}{R} \ , \tag{12}$$

where χ is an arbitrary function and the factor $\theta(ct - R)$ is introduced to satisfy the natural boundary condition for $t < 0$ (causality condition). The function χ is determined by imposing the boundary condition for $R \to 0$. It is natural to assume that the time dependence of Ψ is much slower than its spatial dependence, such that $\Psi \to 0$ for $R \to 0$ and

$$\frac{t}{R} + \frac{\chi(-ct)}{R} = 0 \tag{13}$$

from equation (12). We can see that this condition amounts to assuming that the focal perturbation occurs with a slower velocity than the wave velocity c (it follows $l < cT$, where l is the localization length of the focus, *i.e.* the localization length of the function $\delta(\mathbf{R})$). It follows from equation (13) $\chi(x) = x/c$ as the leading term; we get the solution

$$\Psi = \frac{t}{R} + \frac{1}{c}(1 - \frac{ct}{R})\theta(ct - R) + const = \frac{1}{c}\left[\theta(ct - R) + \frac{ct}{R}\theta(R - ct)\right] + const \ , \tag{14}$$

where $const = -1/c$; up to this constant, this is precisely the Kirchhoff solution

$$\Psi = \frac{1}{4\pi c^2}\int d\mathbf{R}'\frac{\delta(t - \left|\mathbf{R} - \mathbf{R}'\right|/c)}{\left|\mathbf{R} - \mathbf{R}'\right| R'} = \frac{1}{4\pi c^2}\int d\mathbf{r}\frac{\delta(t - r/c)}{r\left|\mathbf{R} - \mathbf{r}\right|} \ ; \tag{15}$$

indeed, the integral in equation (15) gives the θ-functions in equation (14).

The potential Ψ given by equation (14) satisfies the homogeneos equation $\ddot{\Psi} - c^2\Delta\mathbf{\Psi} = 0$, except for $ct = R$, where the solution is not determined (the functions θ are not determined for $R = ct$). It follows that we should disregard contributions to Ψ for $R \neq ct$ and determine the solution in the vicinity of $R = ct$ by other means; this is the regularization (calibration) procedure; the θ-prefactors have not a physical relevance for $R = ct$. It is known that the potentials have not a direct physical relevance; in particular, including the advanced solution in the half-space, the boundary conditions at the surface of the half-space cannot be satisfied with the unregularized solution (Lamb problem[8]-[11]). Similarly, for $m_{ij} = m\delta_{ij}$ the solution of equation (1) is $\mathbf{u} = grad\Phi$, where

$$\Phi = \frac{Tm}{4\pi c_l^2} \int d\mathbf{R}' \frac{\delta(t - \left|\mathbf{R} - \mathbf{R}'\right| / c_l)}{|\mathbf{R} - \mathbf{R}'|} \delta(\mathbf{R}') = \frac{Tm}{4\pi c_l} \frac{\delta(R - c_l t)}{R} \ , \tag{16}$$

while equations (9) and (14) give $\mathbf{u} = -\frac{Tm}{4\pi} grad\Delta\Psi_l$, *i.e.* $\Phi = -\frac{Tm}{4\pi}\Delta\Psi_l$,

$$\Phi = -\frac{Tm}{4\pi}\Delta\Psi_l = \frac{Tm}{2\pi c_l} \frac{\delta(R - c_l t)}{R} \ ; \tag{17}$$

we can see that these two expressions (equations (16) and (17)) differ by a factor 1/2.

Since second-order derivatives (at least) may have relevance (equations (9), (10)), we apply the calibration procedure to expressions like

$$c\partial_i\partial_j\Psi = -\tfrac{\delta_{ij}}{R}(1 - ct/R)\delta + \tfrac{x_i x_j}{R^3}(1 - 3ct/R)\delta - \tfrac{x_i x_j}{R^2}(1 - ct/R)\delta' - \tfrac{ct\delta_{ij}}{R^3}\theta + \tfrac{3ct x_i x_j}{R^5}\theta \tag{18}$$

(where we neglect the argument $R - ct$); hence

$$c\Delta\Psi = -\frac{2}{R}\delta - (1 - ct/R)\delta' \ . \tag{19}$$

If we apply the laplacian to equation (8) we get

$$\frac{\partial^2}{\partial t^2}\Delta\Psi - c^2\Delta(\Delta\mathbf{\Psi}) = -4\pi\delta(t)\delta(\mathbf{R}) \ , \tag{20}$$

whence, with the Kirchhoff solution,

$$\Delta\Psi = -\frac{\delta(R - ct)}{cR} \ ; \tag{21}$$

comparing equation (19) against equation (21) we can see that the δ'-contributions should be neglected in equation (18), as well as the θ-functions, and a factor 1/2 should be inserted; we get the calibrated expression

$$c\partial_i\partial_j\Psi = -\tfrac{\delta_{ij}}{2R}(1 - ct/R)\delta + \tfrac{x_i x_j}{2R^3}(1 - 3ct/R)\delta \ . \tag{22}$$

Making use of equations (9) and (22) we get immediately the displacement $\mathbf{u} = \mathbf{u}^{nf} + \mathbf{u}^{ff}$, where the near-field displacement is

$$\begin{aligned} u_i^{nf} &= -\tfrac{Tm_{ij}x_j}{4\pi c_t R^3}\delta(R - c_t t) + \\ &+ \tfrac{T}{8\pi R^3}\left(m_{jj}x_i + 4m_{ij}x_j - \tfrac{9m_{jk}x_i x_j x_k}{R^2}\right)\left[\tfrac{1}{c_l}\delta(R - c_l t) - \tfrac{1}{c_t}\delta(R - c_t t)\right] \end{aligned} \tag{23}$$

and the far-field displacement is

$$u_i^{ff} = \tfrac{Tm_{ij}x_j}{4\pi c_t R^2}\delta'(R - c_t t) + \tfrac{Tm_{jk}x_i x_j x_k}{4\pi R^4}\left[\tfrac{1}{c_l}\delta'(R - c_l t) - \tfrac{1}{c_t}\delta'(R - c_t t)\right] \ . \tag{24}$$

These are spherical-shell waves. Similarly, using equations (10) and (22) we get the solution

$$\mathbf{u} = \frac{T\mathbf{f}}{4\pi c_t R}\delta(R - c_t t) + \frac{T\mathbf{R}(\mathbf{R}\mathbf{f})}{4\pi R^3}\left[\frac{1}{c_l}\delta(R - c_l t) - \frac{1}{c_t}\delta(R - c_t t)\right] \tag{25}$$

for the Stokes problem.

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