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On the boundary conditions in fluids

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Abstract

The equilibrium of the elastic solids requires boundary conditions which ensure the local fixed position of their bounding surfaces, in accordance with the linear elasticity where the (fixed) geometrical bounding surface coincides practically with the (moving, displaced) "physical" surface; the displacement in linearly elastic solids is small and its variations in space and time are small; these boundary conditions follow from the determined shape of the solids. The fluid motion is more complicated, since the fluids flow and, in general, they have not a determined shape. Consequently, the boundary conditions for the fluid motion are of a larger variety. Two cases are identified for the motion of fluids with a "free" surface, which may be of physical interest. First, for small variations of the displacement the "free" surface of a fluid may flow "along itself", *i.e.* along directions tangential to the surface, while its motion along the direction normal to the surface should be free of force. This observation implies boundary conditions which involve only the normal derivatives of the surface displacement, in contrast with the boundary conditions relevant for elastic solids. Second, for larger displacements (still satisfying the linear elasticity, *i.e.* with small variations) it may appear that the moving surface should be subject to the equation of motion of the fluid; this latter case would be relevant for the presence of the gravitational field. These boundary conditions lead to eigemodes, which, in gravitational field are, approximately, the well-known gravitational waves, while, in the former case (of small displacements) the eigenmodes are lateral waves, *i.e.* waves parallel with the surface; the lateral waves received little attention, though they may lead to an interesting dynamics. It is shown that the gravity waves are poor approximations as propagating waves; they may be viewed as local disturbances produced by external forces.

The internal motion in fluids is more complicated than in solids, since the fluids flow and have not a determined shape. Let a fluid be at rest and in equilibrium with a constant particle concentration n. The fluid is considered ideal, *i.e.* the viscosity and the thermoconductivity are absent; therefore, its motion proceeds at constant entropy. Let $\mathbf{u}(\mathbf{R},t)$ be a displacement field in the fluid, where $\mathbf{R}$ denotes the position and t is the time. For a slight variation in space and time, the displacement $\mathbf{u}$ generates a change $\delta n=-ndiv\mathbf{u}$ in concentration, which satisfies approximately the continuity equation $\partial\delta n/\partial t+div(n\mathbf{v})=0$, where $\mathbf{v}=d\mathbf{u}/dt\simeq\partial\mathbf{u}/\partial t$ is the particle velocity. Since the variations of the disturbances are assumed small, the transport velocity may be neglected, *i.e.* $d\mathbf{u}/dt=\partial\mathbf{u}/\partial t+(\mathbf{v}grad)\mathbf{u}$ is approximated by $\dot{\mathbf{u}}=\partial\mathbf{u}/\partial t$.

All the forces which act in a fluid derive from the gradient of a pressure p (there are not shear forces in a fluid); it follows that the equation of motion of the field $\mathbf{u}$ is

$$\rho\ddot{\mathbf{u}}=-gradp+\mathbf{e}\ \ , \tag{1}$$

where ρ is the mass density and $\mathbf{e}$ is an external field density (also derived from a pressure). Equation (1) is an approximate form of the Euler equation (Euler equation being the Navier-Stokes equation without viscosity). The displacement field offers the opportunity to describe the motion of the fluids by a reduced form of the Euler equation, corresponding to small variations of the displacement field and small velocites. In equation (1) the pressure p varies slowly in space and time and the external field $\mathbf{e}$ is small.

The change in concentration generates a change in pressure $\delta p = (\partial p/\partial n)_s \delta n$, or $\delta p = V(\partial p/\partial V)_s div\mathbf{u}$, by using the particle volume $V = 1/n$, where $\beta = -(1/V(\partial V/\partial p)_s$ is the compresibility at constant entropy s; we get $\delta p = -(1/\beta) div\mathbf{u}$ and equation (1) becomes

$$\rho\ddot{\mathbf{u}} = \frac{1}{\beta} grad \cdot div\mathbf{u} + \mathbf{e} \ , \tag{2}$$

or

$$\ddot{\mathbf{u}} - c^2 grad \cdot div\mathbf{u} = \mathbf{e}/\rho \ , \tag{3}$$

where $c = 1/\sqrt{\rho\beta}$ is the sound velocity. Equation (3) is a reduced form of the Navier-Cauchy equation of the elastic motion of the solids, corresponding only to one Lame elastic modulus $\lambda = 1/\beta$. The reduced stress tensor is

$$\sigma_{ij} = \rho c^2 u_{kk}\delta_{ij} = \lambda div\mathbf{u}\delta_{ij} = -\delta p\delta_{ij} \ , \tag{4}$$

and the reduced Navier-Cauchy equation of motion is

$$\rho\ddot{u}_i = \partial_j\sigma_{ij} + e_i \ , \tag{5}$$

which is equation (3).

Let us integrate equation (5) over the volume V of the fluid bounded by the surface S:

$$\int_V dv\rho\ddot{u}_i = \int_V dv(\partial_j\sigma_{ij} + e_i) \ ; \tag{6}$$

formally, the integral on the right in equation (6) can be tranformed as

$$\int_V dv(\partial_j\sigma_{ij} + e_i) = \int_S dS_j(\sigma_{ij} + p_s\delta_{ij}) = \int_S dS_i(\rho c^2 div\mathbf{u} + p_S) \ , \tag{7}$$

where p_S is the external pressure on the surface, such that $\mathbf{e}\mid_S = gradp_S$ (more rigurously, $\mathbf{e}\mid_S = gradp\mid_S$). It follows that

$$f_S = \rho c^2 div\mathbf{u}\mid_S + p_S \ , \tag{8}$$

could be taken as the net force (pressure) acting perpendicular to the unit area of the surface at every point of the surface. Since the solids have a fixed shape, the condition of equilibrium, *i.e.* the condition of having a fixed shape at every point (of the surface), would be

$$\rho c^2 div\mathbf{u}\mid_S = -p_S \ ; \tag{9}$$

indeed, this is the particular form of the well-known boundary condition[1]

$$n_j\sigma_{ij}\mid_S = -P_i \tag{10}$$

of elastic solids with only compressional stress, where $\mathbf{n}$ is the unit vector normal to the surface, P_i are the components of the external surface forces (acting inwards), *i.e.* $P_i = p_S\delta_{in}$, n being the coordinate along the direction normal to the surface.

For fluids the situation is different. In general, the fluids have not a fixed shape, such that it may appear that we must distinguish between the "geometrical" surface of the fluid and its "physical" surface, *i.e.* the displaced surface. If the surface displacement is large, large variations may appear, which may cast doubts about the validity of the linear elasticity assumed here. Usually, the fluids are confined to containers, which require special boundary conditions at the walls; e.g., the vanishing of the displacement component normal to the wall. Various other situations may require various other boundary conditions.

Let us consider a fluid with a "free" surface S and extending indefinitely along all other directions; we assume that the motion of the fluid is vanishing at infinity (*i.e.* the displacement field and all its derivatives are vanishing at infinity). Let us write equation (3) as

$$\rho\ddot{u}_i = \rho c^2 \partial_i \partial_j u_j + e_i \tag{11}$$

and integrate it over the volume V:

$$\begin{gathered} \int_V dv \rho \ddot{u}_i = \int_V dv (\rho c^2 \partial_i \partial_j u_j + e_i) = \\ = \int_S dS_j (\rho c^2 \partial_i u_j + p_S \delta_{ij}) \ ; \end{gathered} \tag{12}$$

it is reasonable to assume that the fluid surface may move (slide) "freely" along the tangential directions, but, for small displacements, it is fixed along the normal direction (*i.e.*, the fluid preserves its shape). Therefore, the surface force carrying the labels i, $j = n$ in equation (12) should be vanishing; this leads to the boundary condition

$$\rho c^2 \partial_n u_n \mid_S = -p_S \ . \tag{13}$$

This boundary condition looks different from the boundary condition given by equation (9). The difference originates in the integral

$$I_i = \int_V dv \partial_j \sigma_{ij} \tag{14}$$

which can be written in two different ways. First, we have

$$I_i = \int_V dv \partial_j \sigma_{ij} = \rho c^2 \int dS_i div\mathbf{u} \ ; \tag{15}$$

second, we have

$$I_i = \int_V dv \partial_j \sigma_{ij} = \rho c^2 \int dS_j \partial_i u_j \ . \tag{16}$$

In fact, these two integrals are equal, if we realize that, for fluids, whose surface may flow "along itself", the surface integrals of derivatives along directions tangential to the surface are vanishing. Indeed, in equation (15) we are left with

$$I_i = \rho c^2 \int dS_i div\mathbf{u} = \rho c^2 \int dS_i \partial_j u_j = \rho c^2 \int dS \partial_n u_n \tag{17}$$

and from equation (16) we get the same result

$$I_i = \rho c^2 \int dS_j \partial_i u_j = \rho c^2 \int dS \partial_n u_n \ ; \tag{18}$$

we can see that the boundary condition for a "fixed" fluid surface should be indeed equation (13); the condition given by equation (9) is misleading for fluids, since it incorporates spurios terms which give a vanishing contribution.

It follows that for a fluid bounded by a surface "at rest" the equation of motion

$$\ddot{\mathbf{u}} - c^2 grad \cdot div\mathbf{u} = \mathbf{e}/\rho \tag{19}$$

(equation (3)) must be solved with the boundary condition

$$\rho c^2 \partial_n u_n \mid_S = -p_S \tag{20}$$

(equation (13)).

It is worth noting that the boundary condition given by equation (20) is relevant for a surface "at rest", *i.e.* a surface not acted by normal forces (it may suffer static deformations); it may also flow "along itself". In this case the displacement should be small, with small variations, such that the (moving) "physical surface" coincides practically with the (fixed) geometrical surface and the linear elasticity may be used. It may appear that for large deformations the boundary conditions given by equation (20) could not be used anymore, and we we should resort directly to the initial Euler equation (1). Let us take the limit of this equation to the surface S. On the surface S the internal stress may be set equal to zero; there exists an external pressure p_S which gives a force $gradp_S$ on the surface, or, more rigorously, $gradp \mid_S$, with the only non-vanishing n-component $\partial p/\partial n \mid_S$ (we still admit the linear elasticity); in order to have a moving surface we may admit that an additional force may come from the gravitational field, which gives an n-component of the force equal to $-\rho g \partial u_n/\partial n|_S$, where g is the gravitational acceleration; the coordinate n is the coordinate z along the direction perpendicular to the surface; we get from equation (1) the boundary condition

$$\rho \ddot{u}_n \mid_S = \frac{\partial p}{\partial n} \mid_S - \rho g \frac{\partial u_n}{\partial n} \mid_S \ , \tag{21}$$

or

$$\rho \ddot{u}_z \mid_S = \frac{\partial p}{\partial z} \mid_S - \rho g \frac{\partial u_z}{\partial z} \mid_S \ , \tag{22}$$

This would be the boundary condition for a fluid with a surface set in motion by the gravitational field; in order this condition be satisfied we should have, of course, $\rho g u_n \mid_S \gg p_s$. It is also worth noting that, for this case of a moving surface (with large displacement), the condition of small variations of the displacement both in space and time must be preserved.

We illustrate the difference brought about by the boundary conditions for the motion of a fluid occupying a half-space $z < 0$, bounded by a surface $z = 0$. Equation (19) is solved conveniently by introducing the potentials Φ and φ, such that $\mathbf{u} = grad\Phi$ and $\mathbf{e} = grad\varphi$ (it is assumed that the gravitational field brings no contribution to the body forces which determine the motion). Equation (19) becomes

$$\ddot{\Phi} - c^2 \Delta \Phi = \varphi/\rho \ ; \tag{23}$$

using Fourier transforms with respect to both the time and the coordinates parallel with the plane surface, this equation can be written as

$$\frac{d^2\Phi}{dz^2} + \kappa^2 \Phi = -\frac{\varphi}{\rho c^2} \ , \tag{24}$$

where $\kappa^2 = \omega^2/c^2 - k^2$; ω is the frequency, $\mathbf{k}$ is the in-plane wavevector (coordinate $\mathbf{r}$ in $\mathbf{R} = (\mathbf{r}, z)$) and z is the coordinate along the direction perpendicular to the plane surface. The rigid-surface boundary condition given by equation (20) is

$$\Phi^{(2)} = -\frac{p_S}{\rho c^2} \tag{25}$$

and the moving-surface boundary condition given by equation (21) is

$$\omega^2\Phi^{(0)} = -\frac{p_S}{\rho} + g\Phi^{(1)} \ , \tag{26}$$

where $\Phi^{(0)} = \Phi \mid_{z=0}$, $\Phi^{(1)} = d\Phi/dz \mid_{z=0}$ and $\Phi^{(2)} = d^2\Phi/dz^2 \mid_{z=0}$. The body-force potential φ may be viewed as vanishing on the surface ($\varphi^{(0)} = 0$); then, from equation (24) we get $\Phi^{(2)} = -\kappa^2\Phi^{(0)}$ and $\Phi^{(1)} = -i\kappa\Phi^{(0)}$; the boundary conditions given by equations (25) and (26) become

$$\begin{gathered}\kappa^2\Phi^{(0)} = \frac{p_S}{\rho c^2} \ , \\ (\omega^2 + i\kappa g)\Phi^{(0)} = -\frac{p_S}{\rho} \ .\end{gathered} \tag{27}$$

We can see easily that the first equation (27) (rigid surface) yields eigenmodes with frequency $\omega = ck$ ($\kappa^2 = 0$); these are lateral waves which move tangential to the surface; while the second equation (27) (moving surface) yields eigemodes with frequency $\omega = \sqrt{gk}$ for $k \gg g/c^2$; these are gravitational waves; since $g/c^2 \simeq 10^{-5}m$ ($c \simeq 10^3 m/s$), we may admit that these gravitational waves exist (for not very short wavelength). While the gravitational waves are known since long, the lateral waves are receiving little attention; they can be identified both in elastic solids and fluids.

These eigenmodes, in the presence of an external pressure p_s (excited by an external pressure), may lead to a special, particular, new type of motion in fluids (see jtp251).

Unfortunately, the derivation given above for the gravity waves is inconsistent. Indeed, the approximation $\kappa \simeq ik$ in the second equation (27) (which gives a damped wave $\sim e^{kz}$) implies $\kappa^2 + k^2 = 0$, or $\omega = 0$ (in contrast with $\omega = \sqrt{gk}$) and $\Delta\Phi = 0$ (or $div\mathbf{u} = 0$), which means the absence of the internal stress (given by $c^2 grad \cdot div\mathbf{u}$). A displacement $\mathbf{u}$ implies always an internal stress associated with a variation of the pressure, such that the internal stress may not be neglected, even in the presence of the gravitational field. Moreover, the condition $k \gg g/c^2$ required by the approximation described above amounts to $\rho c^2 div\mathbf{u} \gg \rho g u_z$, through $c^2/\lambda \gg g$, which indicates that the internal stress gives higher effects than the gravitational field, even for a small displacement $\mathbf{u}$. It follows that the effect of the gravitational field is small, it may be neglected, and the "fixed"-surface boundary condition $\rho c^2 \partial_n u_n \mid_S = -p_S$ is appropriate. On the other hand, taking the limit of the equation of motion to the bounding surface is not warranted, since external forces occur on the surface, which should compensate the internal forces; it makes the term $\rho\ddot{\mathbf{u}}$ invalid on the surface.

The disturbance produced by the gravitational field can be included in the equation of motion for u_z through

$$\rho\ddot{u}_z - \rho c^2 \partial_z div\mathbf{u} + \rho g \partial_z u_z = e_z \ , \tag{28}$$

or, leaving aside the in-plane contributions,

$$\ddot{u}_z - c^2 u_z^{''} + g u_z^{'} = e_z/\rho \ , \tag{29}$$

where we can see the occurrence of a small friction-like term, which may be eliminated by $u_z \sim e^{gz/2c^2}$.

There exist special cases where we may neglect the internal stress and keep only the gravitational field. For instance, in a shallow canal where all the fluid is displaced along the z-coordinate, we may use the approximaton $g u_z^{'} \simeq -h u_z^{''}$, where h is the depth of the canal; we get the canal waves propagating with velocity $v = \sqrt{gh}$. Similarly, for a disturbance produced by external forces, as for tidal waves, we may set $u_z \sim e^{kz}$, which gives oscillations (not waves!) with frequency $\omega = \sqrt{gk}$; k remaining undetermined. In general, the equation $\ddot{u}_z + gu_z^{'} = e_z$ is a diffusion equation in time. In all cases, when necessary, the "fixed"-surface boundary condition is appropriated. If we set $div\mathbf{u} = 0$, then we have the equations $\rho\ddot{u}_{x,y} = e_{x,y}$, $\rho\ddot{u}_z + \rho g u_z^{'} = e_z$; for $\mathbf{e} = 0$ and $\omega^2 = gk$ we get $u_z \neq 0$ and $u_{x,y} = 0$, which do not satisfy the condition $div\mathbf{u} = 0$.

Another proof of the above point is provided by the equations of motion written as $\rho\partial_t v_x = \rho\partial_t v_y = 0$, $\rho\partial_t v_z = -\partial_z p$; using the velocity potential $\mathbf{v} = grad\psi$, we get $\partial_x\dot{\psi} = \partial_y\dot{\psi} = 0$ and $\rho\dot{\psi} = -p$ (this equation is an integral of the equations of motion); in addition, the equation of continuity requires $div\mathbf{v} = \Delta\psi = 0$, hence $\Delta p = 0$; if we write $p = \rho g u_z$, we should have $\Delta u_z = 0$, or $\Delta v_z = 0$, while from equation $\rho\dot{\psi} = -p = -\rho g u_z$ we get $\ddot{\psi} = -g\partial_z\psi$; hence we may have $\omega^2 = gk$ for $\psi \sim e^{kz}$, but from $\partial_x\dot{\psi} = \partial_y\dot{\psi} = 0$ it follows that ψ does not depend on x and y; therefore, the condition $\Delta\psi = 0$ reduces to $\partial_z^2\psi = 0$, or $k = 0$.

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References

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