Journal of Theoretical Physics

Founded and Edited by M. Apostol **258** (2017)

ISSN 1453-4428

Vibrations of fluids

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Abstract

Vibrations of an ideal fluid are considered for a boundary condition which requires the equlibrium of the free surface (free of forces) along its normal; the fluid surface may flow along itself, withut physical consequences. The vibrations of an ideal fluid confined to a half-space are computed for point volume forces or surface pressure, either oscillating or time impulses. Specific eigenmodes are identified for the half-space fluid, which are lateral waves propagating along directions parallel with the surface. The contribution of these eigenmodes is highlighted, especially for a point time-impulse pressure, where cylindrical waves with a singular wavefront are produced. Similar eigenmodes (lateral waves) are identified for two superposed fluids, each occupying a half-space and separated by a plane membrane (interface). A fluid confined to a rectangular box with two plane-parallel surfaces is also considered, with a finite thickness and a solid (rigid) surface at the bottom, which exhibits discrete vibration modes.

Introduction. The internal motion of the ideal flluids is described by the reduced Euler equation

$$\ddot{\mathbf{u}} - c^2 grad \cdot div\mathbf{u} = \mathbf{e} \ \ , \tag{1}$$

where $\mathbf{u}$ is the displacement field exhibiting small variations in space and time, c is the (adiabatic) sound velocity and $\mathbf{e}$ is an external field (per unit mass). The equation originates in the small variations $\delta n = -ndiv\mathbf{u}$ of the particle concentration n, which generate small variations $\delta p = (\partial p/\partial n)\delta n = -n(\partial p/\partial n)div\mathbf{u}$, or $\delta p = -(1/\beta)div\mathbf{u}$, where β is the (adiabatic) compressibility; consequently, an internal stress $-grad\delta p$ arises in the equation of motion $\rho\ddot{\mathbf{u}} = -grad\delta p$, where ρ is the mass density, which leads to equation (1) with the sound velocity $c = 1/\sqrt{\rho\beta}$. Equation (1) is a reduced form of the Navier-Stokes equation of fluid motion (Euler equation, without viscosity and for small velocities) and the Navier-Cauchy equation of elastic motion of homogeneous isotropic solids with only one Lame elastic modulus $\lambda = 1/\beta$. It is worth noting the pression disturbance $\delta p = -\rho c^2 div\mathbf{u}$.

The external field $\mathbf{e}$ derives from a potential φ, $\mathbf{e} = grad\varphi$, such that the displacement $\mathbf{u}$ can be written as deriving from a potential Φ, $\mathbf{u} = grad\Phi$; equation (1) becomes the wave equation

$$\ddot{\Phi} - c^2\Delta\Phi = \varphi \tag{2}$$

(or $\ddot{\mathbf{u}} - c^2\Delta\mathbf{u} = \mathbf{e}$). For $\varphi = Tm\delta(t)\delta(\mathbf{R})$, where m is a torque (moment of force) divided by density and T is the short duration of the time-impulse the solution of equation (2) in the infinite space (Green function) is given by the retarded Kirchhoff spherical-shell wave

$$\Phi = \frac{Tm}{4\pi c^2}\int d\mathbf{R}'\frac{\delta(t - \left|\mathbf{R} - \mathbf{R}'\right|/c)}{|\mathbf{R} - \mathbf{R}'|}\delta(\mathbf{R}') = \frac{Tm}{4\pi c}\frac{\delta(R - ct)}{R} \ . \tag{3}$$

The propagating solutions are obtained by means of this Green function for any distribution of external fields.

Let us consider the vibrations of a fluid confined to a volume V and bounded by the surface S; they are given by equation (2) with the boundary conditions

$$\rho c^2 \partial_f u_f \mid_S = \rho c^2 \partial_f^2 \Phi \mid_S = -p_s \ \ , \tag{4}$$

where f denotes the coordinate normal to the surface S (u_f is the component of the displacement normal to the surface) and p_s is the external pressure (acting inwards). Indeed, if we integrate equation (1) over the volume V we get

$$\begin{gathered} \int_V dv\ddot{\mathbf{u}} = \int_V dv(c^2 grad \cdot div\mathbf{u} + \tfrac{1}{\rho} gradp) = \\ = \int_S d\mathbf{S}(c^2 div\mathbf{u} + \tfrac{1}{\rho} p_s) \ ; \end{gathered} \tag{5}$$

we can see that the volume integral of $\ddot{\mathbf{u}}$ implies the surface integration over surface normal to $\ddot{\mathbf{u}}$, which means that $\mathbf{u}$ is free to move along directions tangential to the surface; consequently, in $div\mathbf{u}$ in the surface integral it remains only the normal derivative $\partial_f u_f$; the condition of equilibrium requires the vanishing of the total force, *i.e.* equation (4). It is worth noting that this boundary condition holds for a free surface of the fluid with a small displacement (surface at rest); this condition is similar with the condition for solids with determined (fixed) shape, but the boundary condition for solids $n_j \sigma_{ij} \mid_S = -P_i$ differs from equation (4) in that it implies the whole stress tensor $\sigma_{ij} = 2\rho c_t^2 u_{ij} + \rho(c_l^2 - 2c_t^2) div\mathbf{u}\delta_{ij}$, where $u_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$ is the strain tensor, $c_{l,t}$ are the velocities of the elastic waves in solids, $\mathbf{n}$ is the unit vector normal to the surface and $\mathbf{P}$ is the external force acting inwards per unit area of the surface; for $c_t = 0$ (and $c_l = c$) this condition would imply $\rho c^2 div\mathbf{u} = -p_s$ (and $P_\alpha = 0$, where $\alpha = 1, 2$ labels the two tangential coordinates), which differs from equation (4); the difference arises from the fact that the solid surfaces cannot slide along themselves.

Half-space. Let us consider a fluid occupying the half-space $z < 0$ and bounded by the free plane surface $z = 0$. By using Fourier transforms with respect to the time and the coordinates parallel with the surface, equation (2) and the boundary condition (4) become

$$\frac{d^2\Phi}{dz^2} + \kappa^2 \Phi = -\frac{\varphi}{c^2} \ , \ \Phi^{(2)} = -\frac{p_s}{\rho c^2} \ \ , \tag{6}$$

where $\kappa^2 = \omega^2/c^2 - k^2$ and $\Phi^{(2)} = d^2\Phi/dz^2 \mid_{z=0}$, ω being the frequency and $\mathbf{k}$ being the in-plane wavevector. In order to acount conveniently for the boundary condition we multiply equation (6) by $\theta(-z)$ and absorb the step function under the derivative sign; we get

$$\frac{d^2\Phi}{dz^2} + \kappa^2 \Phi = -\frac{\varphi}{c^2} - \Phi^{(1)}\delta(z) - \Phi^{(0)}\delta'(z) \ \ , \tag{7}$$

where $\Phi^{(1)} = d\Phi/dz \mid_{z=0}$ and $\Phi^{(0)} = \Phi_{z=0}$; we limit ourselves to the restriction of the solution Φ to the half-space. Equation (7) is solved by using the Green function $e^{i\kappa|z|}/2i\kappa$; the solution is

$$\Phi = -\frac{1}{2i\kappa c^2} \int_{-\infty}^{0} dz' \left(e^{i\kappa|z-z'|} - e^{i\kappa|z+z'|} \right) \varphi(z') + \Phi^{(0)} e^{i\kappa|z|} \tag{8}$$

and

$$\Phi^{(1)} = -\frac{1}{c^2} \int_{-\infty}^{0} dz' e^{i\kappa|z'|} \varphi(z') - i\kappa\Phi^{(0)} \ ; \tag{9}$$

the boundary condition becomes

$$\kappa^2\Phi^{(0)} = -\varphi^{(0)}/c^2 + p_s/\rho c^2 \ , \tag{10}$$

where $\varphi^{(0)} = \varphi\mid_{z=0}$ (from equation (7) $\Phi^{2)} = -\kappa^2\Phi^{(0)} - \varphi^{(0)}/c^2$); the solution given by equation (8) reads

$$\Phi = -\frac{1}{2i\kappa c^2}\int_{-\infty}^{0} dz^{'}\left(e^{i\kappa|z-z^{'}|} - e^{i\kappa|z+z^{'}|}\right)\varphi(z^{'}) - \frac{\varphi^{(0)}}{c^2\kappa^2}e^{i\kappa|z|} + \frac{p_s}{\rho c^2\kappa^2}e^{i\kappa|z|} \ . \tag{11}$$

We can see the occurrence of the eigenmodes $\omega^2 = c^2k^2$ from $\kappa^2 = 0$ in equation (10), which are lateral waves of the form $e^{-ickt}e^{i\mathbf{kr}}$, where $\mathbf{r}$ is the in-plane position vector.

Let us consider a few particular cases. First, let us take a point source

$$\varphi = m\cos\Omega t\delta(\mathbf{r})\delta(z - z_0) \tag{12}$$

placed at depth z_0 ($z_0 < 0$) and oscillating with frequency Ω; its Fourier transform is

$$\varphi = \pi m[\delta(\omega - \Omega) + \delta(\omega + \Omega)]\delta(z - z_0) \ ; \tag{13}$$

for $p_s = 0$ the solution given by equation (11) is

$$\Phi = -\frac{\pi m}{2i\kappa c^2}\left(e^{i\kappa|z-z_0|} - e^{i\kappa|z+z_0|}\right)[\delta(\omega - \Omega) + \delta(\omega + \Omega)] \tag{14}$$

($\varphi^{(0)} = 0$). Taking the reverse Fourier transforms and using the Sommerfeld-Weyl integral

$$\frac{i}{2\pi}\int d\mathbf{k}\frac{e^{i\mathbf{kr}}}{\kappa}e^{i\kappa|z|} = \frac{e^{i\Omega R/c}}{R} \ , \ R = \sqrt{r^2 + z^2} \ , \tag{15}$$

we get

$$\Phi = \frac{m}{4\pi c^2}\left[\frac{\cos\Omega(t - R_1/c)}{R_1} - \frac{\cos\Omega(t - R_2/c)}{R_2}\right] \ , \tag{16}$$

where $R_{1,2} = \sqrt{r^2 + (z \mp z_0)^2}$; these are monochromatic spherical waves propagating with velocity c, generated by the monochromatic point source, as expected. Since the phases of the two waves are different and depend on position, the superposition of the two waves is in fact a vibration. The displacement at the surface is given by $\mathbf{u}_{\mathbf{r}}^{(0)} = 0$ and

$$u_z^{(0)} = \frac{m\mid z_0\mid\Omega}{2\pi c^3R_0^2}\left[\sin\Omega(t - R_0/c) - \frac{c}{\Omega R_0}\cos\Omega(t - R_0/c)\right] \ , \tag{17}$$

where $R_0 = \sqrt{r^2 + z_0^2}$.

Let us assume a time-impulse wave source

$$\varphi = Tm\delta(t)\delta(\mathbf{r})\delta(z - z_0) \tag{18}$$

with the Fourier transform $\varphi = Tm\delta(z - z_0)$ (and $p_s = 0$); from equation (11) we get the potential

$$\Phi = \frac{Tm}{4\pi c}\left[\frac{\delta(R_1 - ct)}{R_1} - \frac{\delta(R_2 - ct)}{R_2}\right] \ ; \tag{19}$$

the surface displacement is $\mathbf{u}_{\mathbf{r}}^{(0)} = 0$ and

$$u_z^{(0)} = \frac{Tm \mid z_0 \mid}{2\pi c R_0^2}\left[\delta^{'}(R_0 - ct) - \frac{1}{R_0}\delta(R_0 - ct)\right] \; ; \tag{20}$$

we can see in equation (19) the image source placed at $\mathbf{r} = 0$, $z = -z_0$, which generates the reflected waves. We note that the boundary condition, though formally satisfied, is in fact meaningless, since the solution is zero outside the support of the functions δ.

The solution given by equation (11) includes a surface potential

$$\Phi_s = \frac{p_s}{\rho c^2 \kappa^2} e^{i\kappa|z|} \; ; \tag{21}$$

leaving aside the body forces ($\varphi = 0$), we consider a uniform, oscillating external pressure $p_s = p\cos\Omega t$, with the Fourier transform $p_s = \pi p[\delta(\omega - \Omega) + \delta(\omega + \Omega)]$; it gives a surface potential

$$\Phi_s = \frac{p}{4\pi^2 \rho \Omega^2}\cos\Omega(t- \mid z \mid /c) \; ; \tag{22}$$

it is a superposition of two identical travelling waves, propagating in opposite directions. For a uniform temporal impulse $p_s = Tp\delta(t)$ we get

$$\Phi_s = \frac{Tp}{4\pi^2 \rho c}(ct- \mid z \mid)\theta(ct- \mid z \mid) \tag{23}$$

and the displacement $u_z = (Tp/4\pi^2\rho c)\theta(ct- \mid z \mid)$.

Let us consider an oscillating external pressure localized over the small distance d near the origin, given by

$$p_s = d^2 p\delta(\mathbf{r})\cos\Omega t \;\; , \tag{24}$$

with the Fourier transform

$$p_s = \pi p d^2[\delta(\omega - \Omega) + \delta(\omega + \Omega)] \; ; \tag{25}$$

it gives rise to a surface potential

$$\Phi_s = \frac{pd^2}{8\pi^2 \rho}\int d\mathbf{k} e^{i\mathbf{kr}}\int d\omega \frac{e^{i\kappa|z|}e^{-i\omega t}}{\omega^2 - c^2 k^2}[\delta(\omega - \Omega) + \delta(\omega + \Omega)] \; . \tag{26}$$

It is easy to see that the contribution of the δ-functions is zero; it remains only the contribution of the eigenmodes $\omega = \pm ck$ (lateral waves); the integration should be performed over the lower half-plane in order to ensure the causality ($\Phi_s = 0$ for $t < 0$); the result is

$$\Phi_s = -\frac{pd^2}{2\rho c^2}J_0(\Omega r/c)\sin\Omega t \;\; , \tag{27}$$

which are in-plane vibrations governed by the Bessel function of zeroth order. The displacement does not depend on z. It is worth noting that the boundary condition $\Phi^{(2)} = -p_s/\rho c^2$ is not satisfied at the position of the singular external pressure $\mathbf{r} = 0$, as expected.

Now, let us assume an external pressure of the form

$$p_s = Tpd^2\delta(\mathbf{r})\delta(t) \; ; \tag{28}$$

it gives a surface potential

$$\Phi_s = -\frac{Tpd^2}{2\pi\rho c}\frac{1}{\sqrt{c^2t^2 - r^2}}\theta(ct - r) \;\; , \tag{29}$$

where we use the integrals[1]

$$\int_0^\infty dx J_0(x)\cos\lambda x = \frac{1}{\sqrt{1-\lambda^2}}\theta(1-\lambda)\ ,\ \int_0^\infty dx J_0(x)\sin\lambda x = \frac{1}{\sqrt{\lambda^2-1}}\theta(\lambda-1)\ . \tag{30}$$

The waves corresponding to the surface potential given by equation (29) are cylindrical waves with a singular wavefront at $r = ct$.

Two superposed fluids. Let us consider two superposed fluids, one, denoted by 1, occupying the half-space $z < 0$, another, denoted by 2, occupying the half-space $z > 0$, separated by a thin, plane membrane at $z = 0$. All the quantities pertaining to the fluid 1 carry the label 1, all the quantities pertaining to the fluid 2 carry the label 2. The potential $\Phi_1$is given by equation (8), while the potential Φ_2 is given by

$$\Phi_2 = \Phi_2^{(0)} e^{i\kappa_2 z}\ ; \tag{31}$$

the wave source is placed in fluid 1. The boundary conditions ensure the continuity of the nornal displacement at the interface $z = 0$,

$$u_{1z}^{(0)} = u_{2z}^{(0)}\ \ , \tag{32}$$

and the equilibrium of the interface,

$$\rho_1 c_1^2 \Phi_1^{(2)} + \rho_2 c_2^2 \Phi_2^{(2)} = -p_s\ ; \tag{33}$$

these boundary conditions lead to

$$\kappa_1\Phi_1^{(0)} + \kappa_2\Phi_2^{(0)} = -\frac{1}{ic_1^2}\int_{-\infty}^0 dz' e^{i\kappa_1|z'|}\varphi(z') \tag{34}$$

and

$$\lambda_1\kappa_1^2\Phi_1^{(0)} + \lambda_2\kappa_2^2\Phi_2^{(0)} = p_s \tag{35}$$

(where we use $\rho c^2 = \lambda$). The solutions of the equations (34) and (35) are

$$\Phi_1^{(0)} = \frac{1}{\kappa_1(\lambda_1\kappa_1 - \lambda_2\kappa_2)}\left[\frac{\lambda_2\kappa_2}{ic_1^2}\int_{-\infty}^0 dz' e^{i\kappa_1|z'|}\varphi(z') + p_s\right] \tag{36}$$

and

$$\Phi_2^{(0)} = -\frac{1}{\kappa_2(\lambda_1\kappa_1 - \lambda_2\kappa_2)}\left[\frac{\lambda_2\kappa_2}{ic_1^2}\int_{-\infty}^0 dz' e^{i\kappa_1|z'|}\varphi(z') + p_s\right] - \frac{1}{i\kappa_2 c_1^2}\int_{-\infty}^0 dz' e^{i\kappa_1|z'|}\varphi(z')\ . \tag{37}$$

The zeroes $\omega = \pm\omega_0$ of the denominator $\lambda_1\kappa_1 - \lambda_2\kappa_2$, where

$$\omega_0 = \sqrt{\frac{\lambda_1^2 - \lambda_2^2}{\rho_1\lambda_1 - \rho_2\lambda_2}}k\ \ , \tag{38}$$

give the eigenmodes of the two fluids (lateral waves), for $\lambda_1 > \lambda_2$, $\rho_1\lambda_1 > \rho_2\lambda_2$ and $\rho_1\lambda_2 < \rho_2\lambda_1$ (or for interchanged labels). Their contribution is governed by the derivative

$$d = (\lambda_1\kappa_1 - \lambda_2\kappa_2)'\ |_{\omega_0} = (\lambda_1^2 - \lambda_2^2)\sqrt{\frac{\lambda_1^2 - \lambda_2^2}{\lambda_! \lambda_2(\rho_2\lambda_1 - \rho_1\lambda_2)}}\ \ , \tag{39}$$

which gives $\lambda_1\kappa_1 - \lambda_2\kappa_2 \simeq d(\omega - \omega_0) + ...$ and $\lambda_1\kappa_1 - \lambda_2\kappa_2 \simeq -d(\omega + \omega_0) + ...$

Plane-parallel surfaces. Let us consider a fluid confined in a volume with plane-parallel surfaces, with thickness d, bounded by two thin membranes placed at $z=-d$ and $z=0$. If we multiply the wave equation (6) by $\theta(-z)\theta(z+d)$ and absorb the θ-functions in the derivative, equation (6) becomes

$$\Phi^{''}+\kappa^2\Phi=-\varphi/c^2-\Phi_0^{(0)}\delta^{'}(z)+\Phi_{-d}^{(0)}\delta^{'}(z+d)-\Phi_0^{(1)}\delta(z)+\Phi_{-d}^{(1)}\delta(z+d)\ ; \tag{40}$$

the solution can be written by using the Green function $e^{i\kappa|z|}/2i\kappa$, and the parameters $\Phi_0^{(1)}$ and $\Phi_{-d}^{(1)}$ can be eliminated in favour of $\Phi_0^{(0)}$ and $\Phi_{-d}^{(0)}$; they are given by

$$\begin{array}{c}\Phi_0^{(1)}=-\frac{1}{1-x^2}\frac{1}{c^2}\int_{-d}^0 dz^{'}e^{-i\kappa z^{'}}\varphi(z^{'})+\frac{x^2}{1-x^2}\frac{1}{c^2}\int_{-d}^0 dz^{'}e^{i\kappa z^{'}}\varphi(z^{'})-\\ -\frac{1+x^2}{1-x^2}i\kappa\Phi_0^{(0)}+\frac{x}{1-x^2}2i\kappa\Phi_{-d}^{(0)}\ ,\\ \Phi_{-d}^{(1)}=-\frac{x}{1-x^2}\frac{1}{c^2}\int_{-d}^0 dz^{'}e^{-i\kappa z^{'}}\varphi(z^{'})+\frac{x}{1-x^2}\frac{1}{c^2}\int_{-d}^0 dz^{'}e^{i\kappa z^{'}}\varphi(z^{'})-\\ -\frac{x}{1-x^2}2i\kappa\Phi_0^{(0)}+\frac{1+x^2}{1-x^2}i\kappa\Phi_{-d}^{(0)}\ ,\end{array} \tag{41}$$

where $x=e^{i\kappa d}$. Introducing these parameters in the expression of the potential Φ, we get

$$\begin{array}{c}\Phi=-\frac{1}{2i\kappa c^2}\int_{-d}^0 dz^{'}\left(e^{i\kappa|z-z^{'}|}-e^{i\kappa|z+z^{'}|}\right)\varphi(z^{'})+\\ +\frac{x^2}{1-x^2}\frac{1}{2i\kappa c^2}\left(e^{i\kappa z}-e^{-i\kappa z}\right)\int_{-d}^0 dz^{'}\left(e^{i\kappa z^{'}}-e^{-i\kappa z^{'}}\right)\varphi(z^{'})-\\ -\frac{1}{1-x^2}\Phi_0^{(0)}\left(x^2e^{i\kappa z}-e^{-i\kappa z}\right)+\frac{x}{1-x^2}\Phi_{-d}^{(0)}\left(e^{i\kappa z}-e^{-i\kappa z}\right)\ .\end{array} \tag{42}$$

The boundary conditions required by the equilibrium of the two free surfaces (free of forces) are given by

$$\begin{array}{c}\Phi_0^{(0)}=-\varphi_0^{(0)}/c^2\kappa^2+p_0/\rho c^2\kappa^2\ ,\\ \Phi_{-d}^{(0)}=-\varphi_{-d}^{(0)}/c^2\kappa^2+p_{-d}/\rho c^2\kappa^2\end{array} \tag{43}$$

(equation (6)); these boundary conditions give the parameters $\Phi_{0,-d}^{(0)}$, which complete the solution of the problem. We can see that the eingenmodes $\omega=\pm ck$ (lateral waves, $\kappa^2=0$) occur for discrete values $k_n=\pi n/d$, n any integer, given by $x^2=1$.

A realistic boundary condition would require the vanishing of the normal component of the displacement for $z=-d$ (a rigid lower surface, where the fluid is in contact with a solid); this condition reads $u_z^{(0)}(z=-d)=d\Phi/dz\mid_{z=-d}=0$; from equation (42), leaving aside the body forces ($\varphi=0$), it leads to

$$\Phi_{-d}^{(0)}=\frac{2x}{1-x^2}\Phi_0^{(0)} \tag{44}$$

and

$$\Phi=\frac{1}{(1-x^2)^2}\left[x^2(1+x^2)e^{i\kappa z}+(1-3x^2)e^{-i\kappa z}\right]\Phi_0^{(0)}\ , \tag{45}$$

where $\Phi_0^{(0)}=p_0/\rho c^2\kappa^2$ (equations (43)). For $p_0=Tp\delta(t)\delta(\mathbf{r})$ we get

$$\Phi=-\frac{Tp}{4\pi^2\rho c}\int d\mathbf{k}e^{i\mathbf{kr}}\frac{\sin ckt}{k}\frac{1}{1-x^2} \tag{46}$$

and

$$\Phi = -\frac{Tp}{8\rho cd} \sum_n J_0(\pi rn/d) \sin(\pi ctn/d) \ ; \qquad (47)$$

we can see the contribution of the poles $k = k_n$ placed on the real axis (slightly below).

Concluding remarks. Vibrations of an ideal fluid bounded by a free surface are analyzed, by using the boundary condition which relates the external pressure to the normal derivative of the normal component of the displacement. Various cases are considered for the external pressure, including uniform or point pressure, with an oscillating or impulse time dependence. Vibrations are computed for a half-space fluid, two superposed fluids and a fluid with finite thickness, comprised between two plane-parallel surfaces, the lower surface being rigid (with a vanishing normal component of the displacement as boundary condition). Special eigenmodes are identified in theses cases, which are lateral waves propagating along directions parallel with the plane surfaces, not depending on the normal coordinate. For a point, time-impulse external pressure these eigenmodes may generate cylindrical waves with singular wavefronts.

Acknowledgments. The author is indebted to the members of the Laboratory of Theoretical Physics at Magurele-Bucharest for many fruitful discussions. This work has been supported by the Scientific Research Agency of the Romanian Governments through Grants 04-ELI / 2016 (Program 5/5.1/ELI-RO), PN 16 42 01 01 / 2016 and PN (ELI) 16 42 01 05 / 2016.

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