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On the Stokes problem and its tensorial generalization in the theory of the elastic waves

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Abstract

A regularization (calibration) procedure is presented for removing unphysical terms in the potentials used in solving the Stokes problem and its tensorial generalization in the theory of the elastic waves. The solution free of such terms is provided for these problems.

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As it is well known the equation of the elastic waves in a homogeneous isotropic body is

$$\ddot{\mathbf{u}} - c_t^2 \Delta \mathbf{u} - (c_l^2 - c_t^2) grad \cdot div \mathbf{u} = \mathbf{f} \ , \tag{1}$$

where $\mathbf{u}$ is the displacement vector, $c_{l,t}$ are the wave velocities and $\mathbf{f}$ is the force per unit mass.[1] For the Stokes problem[2] the force density is $\mathbf{f} = T\mathbf{F}\delta(t)\delta(\mathbf{R})$, where T is a measure of the duration of the force source, $\mathbf{F}$ is the force divided by density, $\mathbf{R}$ is the position vector, t denotes the time and δ is the Dirac function. Forces which depend on $\delta(\mathbf{R})$ or the derivatives of the function $\delta(\mathbf{R})$ are usually called point forces; while forces which include $\delta(t)$ may be termed elementary forces. For the tensorial generalization of the Stokes problem the force components are $f_i = T\delta(t)m_{ij}\partial_j\delta(\mathbf{R})$, i, $j = 1, 2, 3$, where m_{ij} is a constant tensor and ∂_j is the derivative with respect to the coordinate x_j ($\mathbf{R} = (x_1, x_2, x_3)$); the summation over j is assumed in this expression. It is easy to see that the solution of the wave equation with tensorial force is given by $u_i^T = h_j\partial_j u_i^S$, where $F_i h_j$ is replaced by m_{ij} and $\mathbf{u}^S$ is the solution of the Stokes problem; the parameters h_j correspond to the arms of the F_i-forces in a torque representation of the tensor m_{ij}. The Stokes problem and its tensorial generalization are basic elements in Seismology, for describing the propagation of seismic waves generated by (elementary) point seismic sources;[3]-[7] the tensor m_{ij} is the tensor of the seismic moment (divided by density).

The Stokes problem and its tensorial generalization are solved by means of Helmholtz (or Hertz) potentials.[3, 7] It is known that this approach may be plagued with unphysical contributions, which arise usually from static (or quasi-static) solutions of the wave equations. This can be seen immediately by comparing the solution $\mathbf{u}^T$ for the isotropic case $m_{ij} = m\delta_{ij}$ with the solution $\mathbf{u} = grad\Phi$, $\ddot{\Phi} - c_l^2\Delta\Phi = Tm\delta(t)\delta(\mathbf{R})$, of the wave equation (1), where

$$\Phi = \frac{Tm}{4\pi c_l}\frac{\delta(R - c_l t)}{R} \tag{2}$$

is the well-known spherical wave; as we shall see immediately below, the two expressions differ by a factor 1/2. We provide here a regularization (calibration) procedure for removing the unphysical terms from the potentials used in solving the equation of the elastic waves.

We focus first on the Stokes problem. We adopt the decomposition $\mathbf{u} = grad\Phi + curl\mathbf{A}$ ($div\mathbf{A} = 0$), $\mathbf{f} = grad\phi + curl\mathbf{H}$ ($div\mathbf{H} = 0$) in Helmholtz potentials Φ, ϕ and $\mathbf{A}$, $\mathbf{H}$, where $\Delta\phi = div\mathbf{f}$ and $\Delta\mathbf{H} = -curl\mathbf{f}$; equation (1) is separated in two standard wave equations $\ddot{\Phi} - c_l^2\Delta\Phi = \phi$, $\ddot{\mathbf{A}} - c_t^2\Delta\mathbf{A} = \mathbf{H}$; making use of

$$\phi = -\frac{T}{4\pi}\delta(t)(\mathbf{F}grad)\frac{1}{R} \ , \ \mathbf{H} = \frac{T}{4\pi}\delta(t)curl\left(\frac{\mathbf{F}}{R}\right) \ , \tag{3}$$

we get the Kirchhoff retarded solutions

$$\begin{aligned} \Phi &= -\tfrac{T}{(4\pi c_l)^2}F_i\partial_i\int d\mathbf{R}_1\tfrac{\delta(t-|\mathbf{R}-\mathbf{R}_1|/c_l)}{|\mathbf{R}-\mathbf{R}_1|}\tfrac{1}{R_1} \ , \\ A_i &= \tfrac{T}{(4\pi c_t)^2}\varepsilon_{ijk}F_k\partial_j\int d\mathbf{R}_1\tfrac{\delta(t-|\mathbf{R}-\mathbf{R}_1|/c_t)}{|\mathbf{R}-\mathbf{R}_1|}\tfrac{1}{R_1} \ , \end{aligned} \tag{4}$$

or

$$\begin{aligned} \Phi &= -(\mathbf{F}grad)G_l \ , \ \mathbf{A} = curl(\mathbf{F}G_t) \ , \\ G_{l,t} &= \tfrac{T}{4\pi c_{l,t}}\left[\theta(c_{l,t}t - R) + \tfrac{c_{l,t}t}{R}\theta(R - c_{l,t}t)\right] \ ; \end{aligned} \tag{5}$$

the solution is

$$u_i = -F_i\Delta G_t + F_j\partial_i\partial_j(G_t - G_l) \ . \tag{6}$$

We can see that the solution consists of two parts: spherical waves propagating with velocities $c_{l,t}$, given by δ-functions and derivatives of δ-functions (arising from the derivatives of the θ-functions in equations (5)), and a quasi-static displacement which includes the functions $\theta(R - c_{l,t}t)$ and extends over the distance $\Delta R = (c_l - c_t)t$. The quasi-static contributions, being proportional to third-order derivatives of t/R, are solutions of the homogeneous wave equation, except for $R = c_{l,t}t$, where the solution is not determined (the functions θ are not determined for $R = c_{l,t}t$). Therefore, we should disregard contributions to the potentials for $R \neq c_{l,t}t$ and determine the solution in the vicinity of $R = c_{l,t}t$ by other means; this is a regularization (calibration) procedure used for getting the solution.

We may see the effect of the unphysical terms in potentials by replacing F_i in equations (5) by $m\partial_i$; we get $\mathbf{A} = 0$ and

$$\Phi = \frac{mT}{2\pi c_l}\frac{\delta(R - c_lt)}{R} + \frac{mT}{4\pi c_l}(1 - c_lt/R)\delta^{'}(R - c_lt) = \frac{mT}{2\pi c_l}\frac{\delta(R - c_lt)}{R} \ , \tag{7}$$

while the direct solution of equation (1) for $f_i = Tm\delta(t)\partial_i\delta(\mathbf{R})$ is $\mathbf{u} = grad\Phi$, where Φ is given by equation (2); we can see a discrepancy of a factor 1/2. Since second-order derivatives are relevant, according to equation (6), we apply the calibration procedure to expressions like

$$\begin{aligned} \partial_i\partial_j(4\pi cG/T) &= -\tfrac{\delta_{ij}}{R}(1 - ct/R)\delta + \tfrac{x_ix_j}{R^3}(1 - 3ct/R)\delta - \\ &- \tfrac{x_ix_j}{R^2}(1 - ct/R)\delta' - \tfrac{ct\delta_{ij}}{R^3}\theta + \tfrac{3ctx_ix_j}{R^5}\theta \ , \end{aligned} \tag{8}$$

where we omit for the moment the labels l, t and the argument $R - ct$; comparing equations (2) and (7), we can see that the $\delta^{'}$-contributions should be neglected in equation (8), as well as the θ-functions, and a factor 1/2 should be inserted; we get the calibrated expression

$$\partial_i\partial_j(4\pi cG/T) = -\tfrac{\delta_{ij}}{2R}(1 - ct/R)\delta + \tfrac{x_ix_j}{2R^3}(1 - 3ct/R)\delta \ ; \tag{9}$$

by construction, it is unique. Making use of this regularization procedure we get immediately, from equations (6), the solution of the Stokes problem

$$\mathbf{u}^S = \frac{T\mathbf{F}}{4\pi c_t R}\delta(R-c_t t) + \frac{T\mathbf{R}(\mathbf{RF})}{4\pi R^3}\left[\frac{1}{c_l}\delta(R-c_l t) - \frac{1}{c_t}\delta(R-c_t t)\right] \ ; \tag{10}$$

using $\mathbf{u}^T = h_j \partial_j \mathbf{u}^S$ and replacing $F_i h_j$ by m_{ij}, we get the solution for the tensorial force, written as $\mathbf{u}^T = \mathbf{u}^{nf} + \mathbf{u}^{ff}$, where the near-field displacement is

$$\begin{array}{c} u_i^{nf} = -\frac{Tm_{ij}x_j}{4\pi c_t R^3}\delta(R-c_t t) + \\ +\frac{T}{8\pi R^3}\left(m_{jj}x_i + 4m_{ij}x_j - \frac{9m_{jk}x_i x_j x_k}{R^2}\right)\left[\frac{1}{c_l}\delta(R-c_l t) - \frac{1}{c_t}\delta(R-c_t t)\right] \end{array} \tag{11}$$

and the far-field displacement is

$$u_i^{ff} = \frac{Tm_{ij}x_j}{4\pi c_t R^2}\delta'(R-c_t t) + \frac{Tm_{jk}x_i x_j x_k}{4\pi R^4}\left[\frac{1}{c_l}\delta'(R-c_l t) - \frac{1}{c_t}\delta'(R-c_t t)\right] \ . \tag{12}$$

We can see that these solutions consist of propagating spherical waves (derivatives of the δ-functions included), as expected.

Finally, we add another, more formal, derivation of the regularization procedure. It consists in regarding $\theta(ct-R)$ and $\theta(R-ct)$ in equation (8) as affected by two unknown factors A and, respectively, B for $ct = R$; then, we get from equations (5)

$$\partial_i\partial_j(4\pi cG/T) = -\frac{\delta_{ij}}{R}(A - Bct/R)\delta + \frac{x_i x_j}{R^3}(A - 3Bct/R)\delta - \frac{x_i x_j}{R^2}(A - Bct/R)\delta' \tag{13}$$

(leaving aside the θ-functions) and

$$\Delta(4\pi cG/T) = -\frac{2}{R}A\delta - (A-B)\delta' \ ; \tag{14}$$

comparing this expression with equation (2) (where $4\pi cG/T = \Phi/m$) we get $A = B = 1/2$ and

$$\begin{array}{c} \partial_i\partial_j(4\pi cG/T) = -\frac{\delta_{ij}}{2R}(1-ct/R)\delta + \frac{x_i x_j}{2R^3}(1-3ct/R)\delta - \frac{x_i x_j}{2R^2}(1-ct/R)\delta' = \\ = -\frac{\delta_{ij}}{2R}(1-ct/R)\delta + \frac{x_i x_j}{2R^3}(1-3ct/R)\delta \ , \end{array} \tag{15}$$

which is indeed the calibrated expression given by equation (9).

In conclusion, we highlighted here unphysical terms which may occur in the potentials used for solving the equation of the elastic waves and provided a regularization (calibration) procedure for removing them. The solution free of such undesirable contributions is provided here for the Stokes problem and its generalization to a tensorial force. In the seismological literature the far-field solution given by equation (12) corresponds to the P (l) and S (t) seismic waves.[3, 7]

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