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Secondary elastic waves from the plane surface of a half-space

Bogdan Felix Apostol
Department of Engineering Seismology, Institute of Earth's Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
afelix@theory.nipne.ro

Abstract

The interaction of the primary elastic waves (spherical shells) with the plane surface of a homogeneous isotropic half-space is analzyed. The primary waves are produced by a tensorial point force, known in Seismology as the tensor of the seismic moment. It is shown that the primary waves move on the surface with a higher velocity than the elastic wave velocity. These waves generate additional (secondary) wave sources on the surface, which propagate on the surface and give rise to secondary waves. The secondary waves on the surface are computed here in a simplified model. It is shown that they exhibit a singular wavefront and a long tail, in qualitative agreement with the seismic mai shock recorded on the Earth's surface.

MSC: 35L10; 35Q74; 74J05; 74J10; 74J15; 74J40; 86A15; 86A17

Key words: elastic waves; surface elastic radiation; main shock

In a previous Letter we derived the far-field elastic waves generated by a tensorial point force in a homogeneous isotropic medium.[1] The tensorial point force is considered in the seismological literature as a realistic seismic source;[2, 3] in the equation of the elastic waves

$$\ddot{\mathbf{u}} - c_t^2\Delta\mathbf{u} - (c_l^2 - c_t^2)grad \cdot div\mathbf{u} = \mathbf{f} \ , \tag{1}$$

where $\mathbf{u}$ is the displacement vector, $c_{l,t}$ are the wave velocities and $\mathbf{f}$ is the force per unit mass, the components of the tensorial force are given by $f_i = T\delta(t)m_{ij}\partial_j\delta(\mathbf{R}-\mathbf{R}_0)$, $i,\ j = 1,\ 2,\ 3$, where T is a measure of the duration of the time-impulse source, m_{ij} is the tensor of the seismic moment (divided by density), $\mathbf{R}$ is the position vector, $\mathbf{R}_0$ is the point of localization of the force and δ is the Dirac delta function; t denotes the time, ∂_j is the derivative with respect to the coordinate x_j ($\mathbf{R} = (x_1,\ x_2,\ x_3)$); due to $\delta(\mathbf{R}-\mathbf{R}_0)$ (and the derivatives of this delta function) such forces are termed point forces; due to the factor $\delta(t)$ they may be termed elementary forces. The far-field solution of this equation is given by[1]

$$u_i = \frac{Tm_{ij}x_j}{4\pi c_t R^2}\delta'(R-c_tt) + \frac{Tm_{jk}x_ix_jx_k}{4\pi R^4}\left[\frac{1}{c_l}\delta'(R-c_lt) - \frac{1}{c_t}\delta'(R-c_tt)\right] \ , \tag{2}$$

where $\mathbf{R}$ stands for $\mathbf{R}-\mathbf{R}_0$; we can see that it consists of propagating spherical waves; due to the δ'-factor, they have the aspect of a double shock and look like localized spherical shells. We call them primary waves. In Seismology the l-wave is called P wave (primary), while the t-wave is called S-wave wave (secondary, shear wave).[2]-[6]

We assume an elastic half-space occupying the region $x_3 = z < 0$ and bounded by the plane surface $z = 0$; we assume also the wave source placed at $\mathbf{R}_0 = (0,\ 0,\ z_0)$, $z_0 < 0$, beneath the surface; we are interested in the interaction of the primary waves with this surface. We recognize immediately that this is equivalent with an inhomogeneous boundary problem for the equation of the elastic waves, in a peculiar formulation: since the primary waves are propagating spherical shells they do not act contiunuosly on the entire surface. We expect the primary waves to create new, propagating wave sources on the surface, which, according to the Huygens principle, will give rise to additional waves; we call them secondary waves. Being generated by sources propagating on the surface, they look like a surface elastic radiation. We expect this radiation to account for the main shock occurring on the surface in typical seismic records. We note that, empirically, the main shock is retarded in comparison with the primary waves, and have all the components comparable in magnitude. The determination of the primary waves and the seismic main shock is a fundamental problem in Seismology.[4]-[11] An indirect way of tackling this problem would be to expand the primary waves in Fourier series with respect of the coordinates parallel with the surface, add solutions of the homogeneous equation and impose boundary conditions (usually free boundaries). Unfortunately, the recomposition of the resulting waves implies approximations which obscure the physical content of the problem. This is why we adopt here a more direct approach, which allows the formulation of a simple model, analytically tractable.

The wavefront of the spherical-shell waves given by equation (2) intersects the surface $x_3 = z = 0$ along a circular line defined by $\overline{\mathbf{R}} = (x_1, x_2, -z_0)$, $\overline{R} = (r^2 + z_0^2)^{1/2}$, where $r = (x_1^2 + x_2^2)^{1/2}$ is the distance from the origin (placed on the surface, the epicentre) to the intersection points (we recall that $\mathbf{R}$ and $\overline{\mathbf{R}}$ are in fact $\mathbf{R} - \mathbf{R}_0$ and $\overline{\mathbf{R}} - \mathbf{R}_0$). The radius $\overline{R}$ moves with velocity c, $\overline{R} = ct$, $t >\mid z_0 \mid /c$, and the in-plane radius r moves according to the law $r = \sqrt{\overline{R}^2 - z_0^2} = \sqrt{c^2t^2 - z_0^2}$, where c stands for the velocities $c_{l,t}$; its velocity $v = dr/dt = c^2t/r$ is infinite for $r = 0$ ($\overline{R} = ct =\mid z_0 \mid$) and tends to c for large distances.

The finite duration T of the source makes the δ'-functions in equation (2) to be viewed as functions with a finite spread $l = \Delta R = cT \ll R$; consequently, the intersection line of the waves with the surface has a finite spread Δr, which can be calculated from

$$\overline{R}^2 = r^2 + z_0^2 \quad , \quad (\overline{R} + l)^2 = (r + \Delta r)^2 + z_0^2; \tag{3}$$

hence,

$$\Delta r \simeq \frac{2\overline{R}l}{r + \sqrt{r^2 + 2\overline{R}l}}. \tag{4}$$

we can see that for $r \to 0$ the width $\Delta r \simeq \sqrt{2 \mid z_0 \mid l}$ of the spot on the surface is much larger than the width of the spot for large distances $\Delta r \simeq l$ ($2 \mid z_0 \mid \gg l$). For values of r not too close to the origin (the epicentre) we may use the approximation $\Delta r \simeq \overline{R}l/r$. As long as the spherical wave is fully included in the half-space its total energy E_0 is given by the (constant) energy density integrated over the spherical shell of radius R and thickness l. If the wave intersects the surface of the half-space, its energy E is given by the energy density integrated over the spherical sector which subtends the solid angle $2\pi(1 + \cos\theta)$, where $\cos\theta =\mid z_0 \mid /\overline{R}$ (see Fig.1). It follows $E = \frac{1}{2}E_0 (1+ \mid z_0 \mid /ct)$ for $ct >\mid z_0 \mid$. We can see that the energy of the wave decreases by the amount $E_s = \frac{1}{2}E_0 (1- \mid z_0 \mid /ct)$, $ct >\mid z_0 \mid$. This amount of energy is transferred to the surface, which generates secondary waves, according to the Huygens principle.

In the spot with the width Δr generated on the surface by the far-field primary waves given by equation (2) we may expect a reaction of the (free) surface, such as to compensate the force exerted by the incoming spherical waves. This localized reaction force generates secondary waves, distinct from the incoming, primary spherical waves. The secondary waves can be viewed as waves

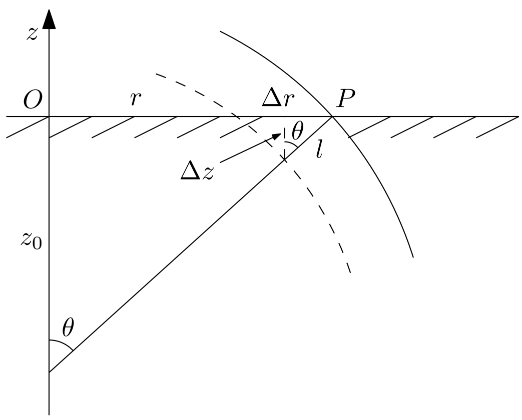

Figure 1: Spherical-shell wave intersecting the surface $z = 0$ at P.

scattered off the surface, from the small region of contact of the surface spot (practically, a circular line). If the reaction force is strictly limited to the zero-thickness surface (as, for instance, a surface force), it would not give rise to waves, since its source has a zero integration measure. We assume that this reaction appears in a surface layer with thickness Δz ($\Delta z \ll \mid z_0 \mid$) and with a surface extension $2\pi r\Delta r$, where it is produced by volume forces. The thickness Δz of the superficial layer activated by the incoming primary wave may depend on $\overline{R}$ (and r), as the surface spread Δr does (equation (4)); for instance, from Fig.1 we have $\Delta z = l \mid z_0 \mid /\overline{R}$. Since for an intermediate, limited region of the variable r (and $\overline{R}$) (*i.e.*, for a region not very close to the origin and not extending to infinity), the dependence on r of the product $\Delta r\Delta z$ is weak, and, in a simplified model, we may neglect this dependence in what follows.

The volume elastic force per unit mass is given by $\partial_j\sigma_{ij}/\rho$, where $\sigma_{ij} = \rho\left[2c_t^2u_{ij} + (c_l^2 - 2c_t^2)u_{kk}\delta_{ij}\right]$ is the stress tensor, u_{ij} is the strain tensor and ρ is the density of the body. The reaction force which compensates this elastic force is

$$f_i = -\partial_j\sigma_{ij}/\rho = -\partial_j\left[2c_t^2u_{ij} + (c_l^2 - 2c_t^2)u_{kk}\delta_{ij}\right] \ . \tag{5}$$

We calculate the strain tensor from the displacement given by equation (2) and use it in equation (5); in order to compute the secondary waves we use the decomposition in Helmholtz potentials. We denote by $\mathbf{u}_s$ the displacement vector in the secondary waves, and introduce the Helmholtz potentials ψ and $\mathbf{B}$ (div$\mathbf{B} = 0$) by $\mathbf{u}_s = \mathrm{grad}\psi + \mathrm{curl}\mathbf{B}$; then, we decompose the force $\mathbf{f}$ according to $\mathbf{f} = \mathrm{grad}\chi + \mathrm{curl}\mathbf{h}$ (div$\mathbf{h} = 0$), where $\Delta\chi = \mathrm{div}\mathbf{f}$ and $\Delta\mathbf{h} = -\mathrm{curl}\mathbf{f}$; by the equation of the elastic waves, the Helmholtz potentials satisfy the wave equations

$$\ddot{\psi} - c_l^2\Delta\psi = \chi \ , \ \ddot{\mathbf{B}} - c_t^2\Delta\mathbf{B} = \mathbf{h} \ ; \tag{6}$$

by straightforward calculations we get $\chi = -c_l^2 div\mathbf{u}$ and $\mathbf{h} = c_t^2\mathrm{curl}\mathbf{u}$, where $\mathbf{u}$ is given by equation (2); therefore,

$$\chi = -\frac{c_lTm_{jk}x_jx_k}{4\pi R^3}\delta''(R - c_lt) \ , \ h_i = \varepsilon_{ijk}\frac{c_tTm_{kl}x_jx_l}{4\pi R^3}\delta''(R - c_tt) \ ; \tag{7}$$

we can see that the potentials χ and $\mathbf{h}$ "move" with velocities c_l and, respectively, c_t (v_l and, respectively, v_t in the plane $z = 0$).

We can calculate the displacement in the secondary waves $\mathbf{u}_s = \mathrm{grad}\psi + \mathrm{curl}\mathbf{B}$, by solving the wave equations (6) with $\chi = -c_l^2u_{ii}$ and $\mathbf{h} = c_t^2\mathrm{curl}\mathbf{u}$ restricted to the superficial layer of thickness Δz and surface spread $2\pi r\Delta r$. Apart from appreciable technical complications, this procedure

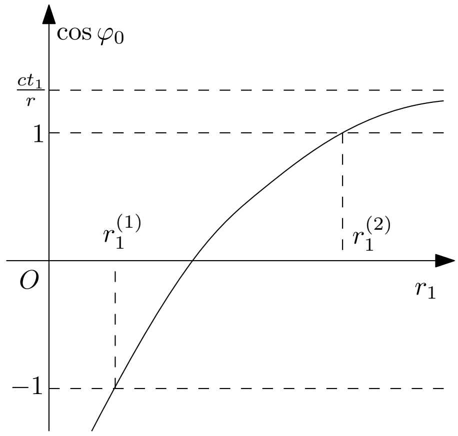

Figure 2: The function $\cos\varphi_0$ *vs* r' for $C > 0$ (equation (16)).

brings many superfluous features which obscure the relevant physical picture. This is why we prefer to use a simplified model which makes use of potentials of the form

$$\chi = \chi_0(r)\delta(z)\delta(r - v_l t) \ , \ \ \mathbf{h} = \mathbf{h}_0(r)\delta(z)\delta(r - v_t t) \tag{8}$$

($div\mathbf{h}_0 = 0$); equations (8) describe wave sources, distributed uniformly along circular lines on the surface, propagating on the surface with constant velocities $v_{l,t}$ and limited to a superficial layer with zero thickness and a circular line with zero width; their magnitudes $\chi_0(r)$ and $\mathbf{h}_0(r)$ have an approximate $1/\overline{R}$-dependence, which has a slow variation for $r \leq\mid z_0 \mid$ (and r not very close to the origin); for this range of the variable r we may consider χ_0 and $\mathbf{h}_0$ as being constant parameters. The velocities $v_{l,t}$ in equation (8) correspond to the velocities $v_{l,t} = dr/dt = c_{l,t}^2 t/r$ calculated above, which are greater than $c_{l,t}$, depend on r and tends to $c_{l,t}$ for large values of the distance r. We make a further simplification and consider them as constant velocities slightly greater than $c_{l,t}$ (over an intermediate, limited range of variation of r). Also, in the subsequent calculations we consider the origin of the time at $r = 0$ (the origin) for each primary wave and the associated secondary source. The simplified model of secondary sources introduced here retains the main features of the exact problem, incorporated in the surface localization and propagation of the sources with velocities $v_{l,t}$ greater than wave velocities $c_{l,t}$; on the other hand, by using this model we lose the anisotropy induced by the tensor m_{ij} and specific details regarding the dependence on the distance. Since the secondary sources are moving sources on the surface we may call the secondary waves produced by these sources "surface elastic (seismic) radiation".

Making use of the potentials given by equation (8), the solutions of equations (6) can be represented as

$$\psi \ = \ \frac{1}{4\pi c_l^2}\int dt_1 \int d\mathbf{R}_1 \frac{\chi_0(r_1)\delta(z_1)\delta(r_1 - v_l t_1)}{|\mathbf{R} - \mathbf{R}_1|}\delta\left(t - t_1 - |\mathbf{R} - \mathbf{R}_1|/c_l\right) \tag{9}$$

and a similar equation for $\mathbf{B}$. First, we focus on the potential ψ, which can be written as

$$\psi \ = \ \frac{1}{4\pi v c^2}\int d\mathbf{r}_1 \frac{\chi_0(r_1)\delta\left[t - r_1/v - \left(r^2 + r_1^2 - 2rr_1\cos\varphi + z^2\right)^{1/2}/c\right]}{\left(r^2 + r_1^2 - 2rr_1\cos\varphi + z^2\right)^{1/2}}, \tag{10}$$

where φ is the angle between the vectors $\mathbf{r}$ and $\mathbf{r}_1$ and we use c and v for c_l and, respectively, v_l, for the sake of simplicity. In order to calculate the integral with respect to the angle φ in equation (10) we introduce the function

$$F(\cos\varphi) \ = \ t - r_1/v - \left(r^2 + r_1^2 - 2rr_1\cos\varphi + z^2\right)^{1/2}/c \tag{11}$$

and look for its zeros, $F_0 = F(\cos\varphi_0) = 0$ $(r_1 < vt)$; we note that, if there exists one root of this equation, there exists another one at least, in view of the symmetry $\cos\varphi = \cos(2\pi - \varphi)$. Then, we expand the function F in a Taylor series in the vicinity of its zero, according to

$$F \;=\; F_0 + (\cos\varphi - \cos\varphi_0)\, F' + ... \;=\; (\cos\varphi - \cos\varphi_0)\, F' + ..., \tag{12}$$

where F' is the derivative of the function F with respect to $\cos\varphi$ for $\cos\varphi = \cos\varphi_0$. It is easy to see that the integral reduces to

$$\psi \;=\; \frac{1}{2\pi cvr}\int_0^\infty dr_1 \frac{\chi_0(r_1)}{\sin\varphi_0}, \tag{13}$$

where φ_0 is the root of the equation $F(\cos\varphi_0) = 0$, lying between 0 and π.

The root $\cos\varphi_0$ is given by

$$F(\cos\varphi_0) \;=\; t - r_1/v - \left(r^2 + r_1^2 - 2rr_1\cos\varphi_0 + z^2\right)^{1/2}/c \;=\; 0, \tag{14}$$

or

$$\left(1 - c^2/v^2\right) r_1^2 - 2\left(r\cos\varphi_0 - c^2t/v\right) r_1 - \left(c^2t^2 - r^2 - z^2\right) \;=\; 0 \tag{15}$$

for $r_1 < vt$. The important feature brought by the diference between the two velocities c and v can be accounted for conveniently by assuming that the two velocities are close to one another; we set $v = c(1+\varepsilon)$, $0 < \varepsilon \ll 1$ (as for sufficiently large distances). In this circumstance we may neglect the quadratic term $\sim r_1^2$ in equation (15) and replace t by the "advanced" time $\tau = t(1-\varepsilon)$ (*i.e.*, $\tau_{l,t} = t(1 - \varepsilon_{l,t})$); we get

$$\cos\varphi_0 \;\simeq\; \frac{2c\tau r_1 - C}{2rr_1} \quad , \quad C \;=\; c^2\tau^2 - r^2 - z^2 \tag{16}$$

for $r_! < vt = c\tau(1+2\varepsilon)$. It is easy to see that this equation has no solution for $C < 0$ (because of the condition $r_! < vt$); for $C > 0$ $(c^2\tau^2 - r^2 - z^2 > 0)$ it has two solutions

$$r_1^{(1)} \;=\; \frac{C}{2(c\tau + r)} \quad , \quad r_1^{(2)} \;=\; \frac{C}{2(c\tau - r)} \tag{17}$$

corresponding to $\cos\varphi_0 = -1$ $(\varphi_0 = \pi)$ and, respectively, $\cos\varphi_0 = 1$ $(\varphi_0 = 0)$ (Fig.2). For $z = 0$ the two roots $r_1^{(1,2)}$ reduce to $r_1^{(1,2)} = (c\tau \mp r)/2$; we can see that the sources of the secondary waves which arrive at r lie inside an anullus with radii $r_1^{(1,2)}$ and a constant width r, which expands on the surface with velocity $c/2$, after a time interval $\tau = r/c$. In the integral given by equation (15) we pass from the variable r_1 to the variable φ_0; for a limited range of integration r (from $r_1^{(1)}$ to $r_1^{(2)}$), we may take χ_0 out of the integral sign; we get

$$\psi \;\simeq\; \frac{C\chi_0}{4\pi c^2}\int_0^\pi d\varphi_0 \frac{1}{(r\cos\varphi_0 - c\tau)^2}, \tag{18}$$

or

$$\psi \;\simeq\; \frac{C\chi_0}{4\pi c^2 r^2}\frac{\partial}{\partial x}\int_0^{\pi/2} d\varphi_0 \left(\frac{1}{\cos\varphi_0 - x} - \frac{1}{\cos\varphi_0 + x}\right) \quad , \quad x = c\tau/r > 1. \tag{19}$$

The integrals in equation (19) can be effected immediately; we get the potential

$$\psi \;\simeq\; \frac{\chi_0}{4c_l^2}\frac{(c_l^2\tau_l^2 - r^2 - z^2)c_l\tau_l}{(c_l^2\tau_l^2 - r^2)^{3/2}}, \tag{20}$$

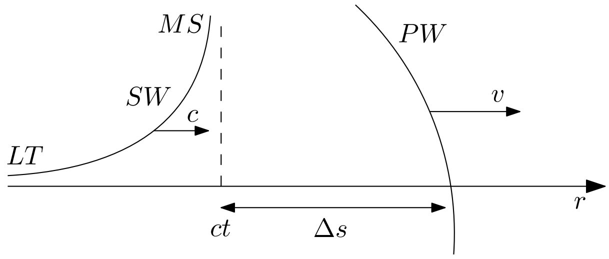

Figure 3: Primary wave (PW), moving with velocity v on the surface, secondary wave (SW), moving with velocity $c < v$, the main shock (MS) and the long tail (LT); the separation between the two wavefronts is $\Delta s = 2(v-c)t$ and the time delay is $\Delta t = (2r/c)(v/c-1)$, where r is the distance on the surface from the origin.

where the velocity c_l is restored. Similarly, we get the vector potential

$$\mathbf{B} \simeq \frac{\mathbf{h}_0}{4c_t^2}\frac{(c_t^2\tau_t^2 - r^2 - z^2)c_t\tau_t}{\left(c_t^2\tau_t^2 - r^2\right)^{3/2}}; \tag{21}$$

these equations are valid for $C_{l,t} = c_{l,t}^2\tau_{l,t}^2 - r^2 - z^2 > 0$.

We can see that the wavefronts $r^2+z^2 = c_{l,t}^2\tau_{l,t}^2$ defines two spherical perturbations which move with velocities $c_{l,t}$. The singular behaviour of these waves for $z = 0$ resembles the algebraic singularity of the waves in two dimensions produced by localized sources.[12, 13] The discontinuities exhibited by these functions are present irrespective of the particular dependence on r of the source potentials, as long as these potentials remain localized; they are related to a constant, finite velocity of propagation of the waves.

Making use of $\mathbf{u}_s = \mathrm{grad}\psi + \mathrm{curl}\mathbf{B}$ we can compute the displacement vector $\mathbf{u}_s$ in the secondary waves. We are interested mainly in the waves propagating on the surface ($z = 0$). First, we note that the displacement is singular at $c_{l,t}\tau_{l,t} = r$; this indicates the existence of two main shocks, occcurring after the arrival of the primary waves. Indeed, the primary waves arrive at the observation point $\mathbf{r}$ at the time $t_p = r/v_{l,t} = (r/c_{l,t})(1-\varepsilon_{l,t})$, while the main shocks occur at $t_m = \tau_{l,t}/(1-\varepsilon_{l,t}) \simeq (r/c_{l,t})(1+\varepsilon_{l,t})$; we can see that there exists a time delay $\Delta t \simeq t_m - t_p \simeq 2(r/c_{l,t})\varepsilon_{l,t}$ between the primary waves and the wavefronts of the secondary waves (the main shocks). The sharp singularity in equations (20) and (21) is related to our using constant velocities $v_{l,t}$; actually, an uncertainty of the form $\Delta v \simeq c\varepsilon$ exists in these velocities, which entails an uncertainty $\tau\varepsilon$ in the time τ, such that the smallest value of the denominator in equations (20) and (21) is of the order $c^2\tau^2\varepsilon$. In the vicinity of the two main shocks the leading contributions to the components of the surface displacement ($z = 0$, in polar cylindrical coordinates) are given by

$$u_{sr} \simeq \frac{\chi_0\tau_l}{4c_l}\cdot\frac{r}{\left(c_l^2\tau_l^2 - r^2\right)^{3/2}} \ , \ u_{s\varphi} \simeq -\frac{h_{0z}\tau_t}{4c_t}\cdot\frac{r}{\left(c_t^2\tau_t^2 - r^2\right)^{3/2}} \tag{22}$$

and

$$u_{sz} \simeq \frac{h_{0\varphi}\tau_t}{4c_t}\cdot\frac{c_t^2\tau_t^2}{r\left(c_t^2\tau_t^2 - r^2\right)^{3/2}} \ ; \tag{23}$$

we can see that there exists a horizontal component of the displacement perpendicular to the propagation direction ($u_{s\varphi}$) and both the r-component and the φ, z-components, which make right angles with the propagation direction, are of the same order of magnitude.[10] For long times ($c_{l,t}\tau_{l,t} \gg r$) the displacement (from equations (22) and (23)) goes like

$$u_{sr} \simeq \frac{\chi_0 r}{4c_l^4\tau_l^2} \ , \ u_{s\varphi} \simeq -\frac{h_{0z}r}{4c_t^4\tau_t^2} \ , \ u_{sz} \simeq \frac{h_{0\varphi}}{4c_t^2 r} \tag{24}$$

which show that the displacement exhibits a long tail, especially the z-component; it subsides as a consequence of the time-dependence induced in the potential $\mathbf{h}_0$ by the integration variable r_1, a circumstance which is neglected in the calculations presented here. The main shock and its long tail obtained here are in qualitative agreement with the seismic records.[3, 9, 10] Primary and secondary waves, the main shock and the long tail are shown in Fig.3.

In conclusion, we have examined in this Letter the interaction of the primary elastic waves with the plane surface of a homogeneous isotropic half-space. The primary waves are propagating spherical shells generated by tensorial point forces, which, in Seismology, are governed by the tensor of the seismic moment. The circle of intersection of the primary waves with the surface expands on the surface with a greater velocity than the velocity of the elastic waves. The primary waves generate additional wave sources, moving on the surface. These sources generate secondary waves, wich may be viewed as surface elastic (seismic) radiation. The secondary waves are estimated in this Letter in a simplified model. It is shown that the secondary waves have a singular wavefront on the surface, which is delayed with respect to the primary waves, known in Seismology as the seismic main shock.

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