Journal of Theoretical Physics

Founded and Edited by M. Apostol **265** (2017)

ISSN 1453-4428

Of longitudinal fields in Electromagnetism

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Abstract

The transverse electromagnetic field generated by an oscillating dipole in the far-field region are reviewed. The longitudinal uniform oscillating electric and magnetic fields inside a capacitor with an oscillating charge and, respectively, inside a solenoid with an oscillating current are computed.

The electromagnetic fields are generated by charges in motion. In this respect the basic element is the oscillating dipole, with the charge and current densities

$$\rho = q\delta(\mathbf{r}-\mathbf{r}_0) \ , \ \mathbf{j} = q\mathbf{v}\delta(\mathbf{r}-\mathbf{r}_0) \ , \ \mathbf{r}_0 = \mathbf{a}\cos\omega t \ \ , \tag{1}$$

where $\mathbf{v} = d\mathbf{r}_0/dt$ is the velocity and ω denotes the frequency; the oscillation amplitude a is much smaller than the distances of interest ($a \ll r$). With usual notations, the Maxwell equations $div\mathbf{E} = 4\pi\rho$, $div\mathbf{H} = 0$, $curl\mathbf{E} = -(1/c)\partial\mathbf{H}/\partial t$, $curl\mathbf{H} = (1/c)\partial\mathbf{E}/\partial t + (4\pi/c)\mathbf{j}$ are solved by $\mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t - grad\Phi$, $\mathbf{H} = curl\mathbf{A}$, with the Lorenz gauge $(1/c)\partial\Phi/\partial t + div\mathbf{A} = 0$; from Maxwell equations the potentials Φ and $\mathbf{A}$ satisfy the wave equations $\ddot{\Phi}/c^2 - \Delta\Phi = 4\pi\rho$, $\ddot{\mathbf{A}}/c^2 - \Delta\mathbf{A} = (4\pi/c)\mathbf{j}$, with the Kirchhoff solutions

$$\Phi = \int d\mathbf{r}' \frac{\rho(\mathbf{r}', t- \mid \mathbf{r}-\mathbf{r}' \mid /c}{\mid \mathbf{r}-\mathbf{r}' \mid} \ , \ \mathbf{A} = \frac{1}{c}\int d\mathbf{r}' \frac{\mathbf{j}(\mathbf{r}', t- \mid \mathbf{r}-\mathbf{r}' \mid /c}{\mid \mathbf{r}-\mathbf{r}' \mid} \ . \tag{2}$$

Beside $a \ll r$ we assume also $a\omega/c \ll 1$, *i.e.* the wavelength c/ω be much longer than the dipole size (it is equivalent with $v \ll c$); then,

$$\mathbf{A} \simeq \frac{q\mathbf{v}(t-r/c)}{cr} \ . \tag{3}$$

The scalar potential is obtained most conveniently from the gauge condition

$$\Phi \simeq \frac{q\mathbf{vr}}{cr^2} + \frac{q\mathbf{r}_0\mathbf{r}}{r^3} \tag{4}$$

(the velocity $\mathbf{v}$ and the position $\mathbf{r}_0$ are taken at the retarded time $t - r/c$). The fields can be decomposed in near fields (nf) and far fields (ff):

$$\begin{gathered}\mathbf{E}^{nf} = -\frac{q\mathbf{v}}{cr^2} + \frac{2q(\mathbf{vr})\mathbf{r}}{cr^4} - \frac{q\mathbf{r}_0}{r^3} + \frac{3q(\mathbf{r}_0\mathbf{r})\mathbf{r}}{r^5} \quad \mathbf{H}^{nf} = \frac{q\mathbf{v}\times\mathbf{r}}{cr^3} \ , \\ \mathbf{E}^{ff} = -\frac{q\dot{\mathbf{v}}}{c^2r} + \frac{q(\dot{\mathbf{v}}\mathbf{r})\mathbf{r}}{c^2r^3} = \frac{q[(\dot{\mathbf{v}}\times\mathbf{r})\times\mathbf{r}]}{c^2r^3} = \mathbf{H}^{ff}\times\frac{\mathbf{r}}{r} \ , \ \mathbf{H}^{ff} = \frac{q\dot{\mathbf{v}}\times\mathbf{r}}{c^2r^2} = \frac{\mathbf{r}}{r}\times\mathbf{E}^{ff} \ ;\end{gathered} \tag{5}$$

the far fields go at infinity $\mathbf{r} \to \infty$ like $1/r$, while the near fields go faster than $1/r$. For $\omega r/c \ll 1$ (*i.e.*, for distances much smaller than the wavelength) the main contribution is brought by the near fields; the electric field in the near-field zone have a longitudinal component (parallel with $\mathbf{r}$). For $\omega r/c \gg 1$, *i.e.* for distances much larger than the wavelength, the main contribution arises from the far fields (this is called the dipole approximation, or the wave-zone approximation); the far fields are transverse, *i.e.* they are perpendicular to $\mathbf{r}$; in addition, they are also perpendicular to each other. These are the fields of the electromagnetic radiation. For a limited region much smaller than the wavelength the radiation fields may be viewed as uniform, oscillating fields; both electric and magnetic fields are present in such a region; they may be represented as the real part of $\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}$, $\mathbf{H} = \mathbf{H}_0 e^{-i\omega t}$ ($\mathbf{E}_0$ and $\mathbf{H}_0$ equal).

An interesting situation occurs for a capacitor, consisting of two infinite plane-parallel plates placed at $z = \pm d/2$, each with the surface charge desnity q_s, which oscillates in time; the charge and current densities are

$$\begin{aligned} \rho &= q_s[\delta(z+d/2) - \delta(z-d/2)]\cos\omega t \ , \\ j_z &= \omega q_s[\theta(z+d/2) - \theta(z-d/2)]\sin\omega t \ . \end{aligned} \tag{6}$$

From equations (2) the scalar potential can be written as

$$\Phi = F(z+d/2) - F(z-d/2) \ , \tag{7}$$

where

$$F(z) = q_s Re\left\{ e^{i\omega t} \int d\mathbf{r}' \frac{e^{-i\frac{\omega}{c}\sqrt{(\mathbf{r}-\mathbf{r}')^2+z^2}}}{\sqrt{(\mathbf{r}-\mathbf{r}')^2+z^2}} \right\} \ . \tag{8}$$

The integration over $\mathbf{r}'$ is extended from $r' = 0$ to $r' = \infty$; the integral in equation (8) can then be effected immediately, with the result

$$F(z) = \frac{2\pi c}{\omega} q_s \sin\omega(t- \mid z \mid /c) \ ; \tag{9}$$

we get

$$\Phi \simeq \begin{cases} -\frac{4\pi c}{\omega} q_s \cos\omega(t-d/2c)\sin\omega z/c \ , \ \mid z \mid < d/2 \ , \\ -\frac{4\pi c}{\omega} q_s sgn(z)\cos\omega(t- \mid z \mid /c)\sin\omega d/2c \ , \ \mid z \mid > d/2 \ ; \end{cases} \tag{10}$$

for $\omega d/c \ll 1$ the scalar potential reduces to

$$\Phi \simeq -4\pi q_s z\cos\omega t \ , \ \mid z \mid < d/2 \tag{11}$$

and zero in the exterior of the capacitor; from the gauge condition we can see that the vector potential may be neglected in this case (and the magnetic field). Therefore, we get inside the capacitor a longitudinal uniform oscillating electric field; $E_z = 4\pi q_s \cos\omega t$, as expected. It is worth noting that the fields are zero for $\omega d/c > 1$, (for $F(z)$ the vector potential is $A_z = 2\pi(cq_s/\omega)sgn(z)\sin\omega(t- \mid z \mid /c) = sgn(z)F(z)$, $z \neq 0$, and $E_z = -(1/c)\dot{A}_z - \partial F/\partial z = 0$, $\mathbf{H} = 0$).

Let us assume an infinitely long circular solenoid with radius a and a uniform, oscillating surface current density $j_\varphi = j_s\delta(r-a)\cos\omega t$; $j_\varphi$is the φ-component (tangential component) of the current density in cylindrical cordinates. We can see that $div\mathbf{j} = 0$ and, by continuity equation, the charge density ρ is zero. Therefore, we have only the vector potential $\mathbf{A}$. It is easy to see, from Kirchhoff formula, that $div\mathbf{A}=0$; it follows that $\mathbf{A}\simeq(-y,\,x,\,0)f(r)$, where $f(r)\rightarrow const$ for $r\rightarrow 0$ and $f(r)\rightarrow 0$ for $r\rightarrow\infty$; the magnetic field is uniform and longitudinal (H_z) near the axis of the solenoid and is vanishing for large distances from the axis in the exterior region. Indeed, for the Kirchhoff formulae it is necessary to use the cartesian cordinates, *i.e.*

$$\mathbf{j}=j_s(-\sin\varphi,\,\cos\varphi,\,0)\delta(r-a)\cos\omega t\ ; \tag{12}$$

the vector potential is given by

$$\mathbf{A}=\frac{2aj_s}{c}(-\sin\varphi,\,\cos\varphi,\,0)Re\left\{e^{-i\omega t}\int_0^{\varphi}d\varphi'\cos\varphi'\int dz\frac{e^{i\frac{\omega}{c}\sqrt{a^2+r^2-2ar\cos\varphi'+z^2}}}{\sqrt{a^2+r^2-2ar\cos\varphi'+z^2}}\right\}\ ; \tag{13}$$

the integral with respect to z is the Hankel function,

$$\int dz\frac{e^{i\frac{\omega}{c}\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}}=i\pi H_0^{(1)}(\omega r/c)=\begin{cases} i\sqrt{\frac{2\pi}{\omega r/c}}e^{i(\omega r/c-\pi/4)}\ ,\ \omega r/c\gg 1\ ,\\ \\ -2\ln(\omega r/c)\ ,\ \omega r/c\ll 1\ ;\end{cases} \tag{14}$$

we get

$$\mathbf{A}=\begin{cases}\pi\sqrt{2\pi}\frac{aj_s}{cr}\frac{\omega a/c}{\sqrt{\omega r/c}}(-y,\,x\,0)\cos[\omega(t-r/c)-\pi/4]\ ,\ r>a\ ,\ \omega r/c\gg 1\ ,\\ \\ \frac{2\pi j_s}{c}\frac{a^2}{a^2+r^2}(-y,\,x\,0)\cos\omega t\ ,\ r<a\ ,\ \omega a/c\ll 1\ .\end{cases} \tag{15}$$

The magnetic field inside the solenoid is $H_z=(4\pi a^2j_s/c)/(a^2+r^2)$. For $\omega a/c>1$ there exists a non-unifirm oscillating longitudinal magnetic field.

Acknowledgments. The author is indebted to his friend S. Stoica for bringing this problem to his attention. Also, the author is indebted to the members of the Laboratory of Theoretical Physics at Magurele-Bucharest for many fruitful discussions. This work has been supported by the Scientific Research Agency of the Romanian Government through Grants 04-ELI / 2016 (Program 5/5.1/ELI-RO), PN 16 42 01 01 / 2016 and PN (ELI) 16 42 01 05 / 2016