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The inverse problem in Seismology. Seismic moment and energy of earthquakes

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Abstract

The inverse problem in Seismology is tackled in this paper under three particular circumstances. First, the inverse problem is defined as the determination of the seismic-moment tensor from the far-field seismic waves (P and S waves). These waves provide the only directly accessible (measurable) experimental data on the earthquakes. We use the analytical expression of the seismic waves determined in a previous work of the author; these waves are given by the solution of the equation of the elastic waves in a homogeneous isotropic body with a seismic-moment source of tensorial forces; the source is localized both spatially and temporally. The far-field waves provide three equations for the sixth unknown parameters of the general tensor of the seismic moment. Second, the Kostrov vectorial (dyadic) representation of the seismic moment is used. This representation relates the seismic moment to the focal displacement in the fault and the orientation of the fault (moment-displacement relation); it reduces the seismic moment to four unknown parameters. Third, the fourth missing equation is derived from the energy of the far-field waves and the mechanical work done by forces in the focal region. In particular, this relation provides access to the focal volume of the fault and the near-field seismic waves. The four equations derived here are solved, and the seismic moment determined, thus solving the inverse problem in the conditions described above. It turns out that the seismic moment is traceless, its magnitude is of the order of the elastic energy stored in the focal region (as expected), and the solution is governed by the unit quadratic from associated to the tensor (related to the magnitude of the longitudinal displacement in the P wave). It is shown that a useful picture of the seismic moment is the conic represented by the associated quadratic form, which is a hyperbola with an arbitrary orientation in space. This hyperbola provides an image for the focal region; its asymptotics are oriented along the focal displacement and the normal to the fault. The eigenvalues and the eigenvectors of the associated quadratic form are calculated. Also, it is shown that the far-field seismic waves allow an estimation of the volume of the focal region, focal strain, duration of the earthquake and earthquake energy; the later quantity is a direct measure of the magnitude of the seismic moment. The special case of an isotropic seismic moment is presented.

Introduction. The inverse problem in Seismology aims at getting information about the nature and structure of the forces acting in the earthquake's focus from measurements of the seismic waves at distances far away from the earthquake focus (at Earth's surface). We present here a solution to this problem by means of the seismic waves derived previously in a homogeneous isotropic body with localized tensorial forces, the Kostrov vectorial representation of the seismic moment for a fault (moment-displacement relation) and the relation between the energy of the earthquakes and the volume of the focal region. This relation is derived by equating the energy carried by the far-field seismic waves to the mechanical work done by forces in the focal region.

The seismic moment and seismic energy are basic concepts in the theory of earthquakes.[1]-[4] The seismic moment has emerged gradually in the first half of the $20th$ century, the first estimation of a seismic moment being done by Aki in 1966.[5] The relations between the seismic moment, seismic energy, the mean displacement in the focal region, the rate of the seismic slip and the earthquake magnitude are recognized today as very convenient tools for characterizing the earthquakes.[6]-[8]

It is well known that typical tectonic earthquakes originate in a localized focal region, with dimensions much shorter than the distance to the observation point (and the seismic wavelengths). Recently, the tensorial seismic force density

$$F_i = M_{ij}\partial_j\delta(\mathbf{R}-\mathbf{R}_0) \tag{1}$$

has been introduced,[9]-[11] where M_{ij} is the tensor of the seismic moment, δ is the Dirac delta function and $\mathbf{R}_0$ is the position of the focus (it can be determined by recording the longitudinal (P) seismic waves); the labels i, j denote the Cartesian axes and summation over repeating suffixes is assumed (throughout this paper). The seismic tensor M_{ij} is a symmetric tensor, which, in general, has six independent components. The force given by equation (1) is a generalization of the double-couple representation of the seismic force; indeed, let us assume a force density $\mathbf{F}(\mathbf{R}) = \mathbf{f}g(\mathbf{R})$, where $\mathbf{f}$ is the force and $g(\mathbf{R})$ is a distribution function; a point couple associated with a force acting along the i-th direction can be represented as

$$f_i g(x_1+h_1, x_2+h_2, x_3+h_3) - f_i g(x_1, x_2, x_3) \simeq f_i h_j \partial_j g(x_1, x_2, x_3) \ , \tag{2}$$

where h_j, $j = 1, 2, 3$, are the components of an infinitesimal displacement $\mathbf{h}$; x_i, $i = 1, 2, 3$, are the coordinates of the position $\mathbf{R}$ and ∂_j denotes the derivative with respect to x_j. The force moment (torque) $t_{ij} = f_i h_j$ is generalized in equation (2) to a symmetric tensor M_{ij}, which is the seismic moment entering equation (1); in addition, the distribution $g(\mathbf{R})$ can be replaced by $\delta(\mathbf{R}-\mathbf{R}_0)$ for a spatially localized focal region. The δ-function used in equation (1) is an approximation for the shape of the focal region. In equation (1) the focus is viewed as being localized over a distance of order l (volume of order l^3), much shorter than the distance R to the observation point ($l \ll R$).

The seismic moment depends on the time t; we may write $M_{ij}(t) = M_{ij}h(t)$, where $h(t)$ is a positive, localized function, which includes the time dependence of the seismic moment; we assume $max[h(t)] = 1$ and denote by T the (short) duration of the seismic event; the time T is much shorter than any time of interest, such that we may view the function $h(t)$ as being represented by $T\delta(t)$. (The function $h(t)$ should not be mistaken for the magnitude of the displacement vector $\mathbf{h}$ used above).

For a homogeneous isotropic body the seismic waves generated by the tensorial force given by equation (1) are governed by the equation of the elastic waves

$$\ddot{u}_i - c_t^2\Delta u_i - (c_l^2 - c_t^2)\partial_i div\mathbf{u} = \frac{1}{\rho}M_{ij}(t)\partial_j\delta(\mathbf{R}) \ , \tag{3}$$

where u_i are the components of the displacement vector $\mathbf{u}$, $c_{l,t}$ are the velocities of the longitudinal and tranverse waves, respectively, ρ is the density and $\mathbf{R}$ is the position vector drawn from the focus (taken as the origin of the reference frame) to the observation point. The solution of this equation has been given in Refs. [9]-[11]; it can be written as $\mathbf{u} = \mathbf{u}^n + \mathbf{u}^f$, where

$$\begin{aligned} u_i^n = & -\tfrac{1}{4\pi\rho c_t^2}\tfrac{M_{ij}x_j}{R^3}h(t-R/c_t)+ \\ & +\tfrac{1}{8\pi\rho R^3}\left(M_{jj}x_i + 4M_{ij}x_j - \tfrac{9M_{jk}x_ix_jx_k}{R^2}\right)\left[\tfrac{1}{c_l^2}h(t-R/c_l) - \tfrac{1}{c_t^2}h(t-R/c_t)\right] \end{aligned} \tag{4}$$

is the near-field displacement (R comparable with l) and

$$u_i^f = -\frac{1}{4\pi\rho c_t^3}\frac{M_{ij}x_j}{R^2}h^{'}(t-R/c_t) - \frac{1}{4\pi\rho}\frac{M_{jk}x_ix_jx_k}{R^4}\cdot\left[\frac{1}{c_l^3}h^{'}(t-R/c_l) - \frac{1}{c_t^3}h^{'}(t-R/c_t)\right] \tag{5}$$

is the far-field displacement ($R \gg l$). The near-field region is defined by distances R of the order l, while the far-field region is defined by distances R much larger than l. The short duration T of the seismic event ((duration of activity of the focus) enters equations (4) and (5) through the derivative $h^{'}(t)$, which is of the order $1/T$. The displacement vectors given by equations (4) and (5) include the longitudinal wave (denoted by suffix l, not to be confused with length l), propagating with velocity c_l, and the transverse wave (suffix t), propagating with velocity c_t; in the far-field region the displacement vector of the longitudinal wave (P wave) and the displacement vector of the transverse wave (S wave) are orthogonal (this is not so for the l, t-waves in the near-field region). As long as the function $h(t)$ may be viewed as a localized function, the magnitude of the displacement vectors varies as $1/R^2$ for the near-field wave and $1/R$ for the far-field waves. Their direction is determined by the tensor of the seismic moment M_{ij} (in particular the vector with components $M_{ij}x_j$). The particular case $h(t) = T\delta(t)$ is called an elementary earthquake in Refs. [9]-[11]. A superposition of forces given by equation (1), localized at different positions $\mathbf{R}_0$ and different times, corresponds to a structured focus, and the elementary displacement given by equations (4) and (5) gives access to the structure factor of the focal region.[9]-[11]

Far-field seismic waves. It is convenient to introduce the notations

$$M_i = M_{ij}n_j \ , \ M_0 = M_{ii} \ , \ M_4 = M_{ij}n_in_j \ , \tag{6}$$

where $\mathbf{n}$ is the unit vector along the radius drawn from the focus to the observation point (observation radius), $x_i = Rn_i$, and $h_{l,t} = h(t - R/c_{l,t})$; henceforth we consider the unit vector $\mathbf{n}$ a known vector. M_0 is the trace of the seismic-moment tensor and M_4 is the quadratic form associated to the seismic-moment tensor, constructed with the unit vector $\mathbf{n}$; we call it the unit quadratic form of the tensor. The vector $\mathbf{M}$ can be called the projection of the tensor along the focus-observation point direction (observation direction).

Making use of these notations, the seismic waves given by equations (4) and (5) can be decomposed into l- and t-waves, written as

$$\begin{gathered} \mathbf{u}^n = \mathbf{u}_l^n + \mathbf{u}_t^n \ , \\ \mathbf{u}_l^n = \frac{h_l}{8\pi\rho c_l^2 R^2}\left[(M_0 - 9M_4)\mathbf{n} + 4\mathbf{M}\right] \ , \\ \mathbf{u}_t^n = -\frac{h_t}{8\pi\rho c_t^2 R^2}\left[(M_0 - 9M_4)\mathbf{n} + 6\mathbf{M}\right] \ , \end{gathered} \tag{7}$$

and

$$\begin{gathered} \mathbf{u}^f = \mathbf{u}_l^f + \mathbf{u}_t^f \ , \\ \mathbf{u}_l^f = -\frac{h_l^{'}}{4\pi\rho c_l^3 R}M_4\mathbf{n} \ , \ \mathbf{u}_t^f = \frac{h_t^{'}}{4\pi\rho c_t^3 R}\left(M_4\mathbf{n} - \mathbf{M}\right) \ . \end{gathered} \tag{8}$$

For numerical purposes we take the "maximum deviation" of the near-field diplacement $\mathbf{u}_{l,t}^n$ (with its sign) at $t = R/c_{l,t}$, *i.e.* for $h_{l,t}(0) = 1$. Equally well, we can take the mean values of the vectors $\mathbf{u}_{l,t}^n$ over the support T of the functions $h_{l,t}$, or ΔR, which is of the order $c_{l,t}T$. Henceforth, $h_{l,t}$ in equations (7) are understood as $h_{l,t}(0) = 1$. The functions $h_{l,t}^{'}$ are scissor-like functions ("double-shock" functions), with two sides with opposite signs, extending over T, or the distance ΔR; their "maximum deviations" are of the order $\pm 1/T$; for numerical estimations it is convenient to introduce the notations $\mathbf{v}_{l,t} = \mathbf{u}^f_{l,t}/Th'_{l,t}$ and take the "maximum deviation" of these functions (with their sign), on any side of the functions $h'_{l,t}$, the same side for $\mathbf{v}_l$ and $\mathbf{v}_t$ ($\mathbf{v}_{l,t}$ may depend on the side of the functions $h'_{l,t}$). Similarly, we can take the mean values of $\mathbf{v}_{l,t}$ over any side of the functions $h'_{l,t}$ (the same for $\mathbf{v_l}$ and $\mathbf{v}_t$). The displacement vectors $\mathbf{v}_{l,t}$ are directly accessible experimentally. Making use of these notations, equations (8) become

$$\mathbf{v}_l = -\frac{1}{4\pi\rho T c_l^3 R} M_4 \mathbf{n} \ , \ \mathbf{v}_t = \frac{1}{4\pi\rho T c_t^3 R} \left(M_4 \mathbf{n} - \mathbf{M} \right) \ . \tag{9}$$

We note that the vectors $R^2\mathbf{u}^n_{l,t}$ and $R\mathbf{v}_{l,t}$ depend on the density ρ, the duration T, the seismic moment and the elastic coefficients of the body (velocities of the elastic waves); if local deviations from this pattern are observed, the body is not locally homogeneous and isotropic.

The displacement in the far-field waves is determined by three independent parameters: the magnitude of the vectors $\mathbf{v}_{l,t}$ (two parameters) and the direction of the transverse vector $\mathbf{v}_t$ (one parameter). Consequently, we may view the equations

$$\mathbf{M} = -4\pi\rho T R \left(c_l^3 \mathbf{v}_l + c_t^3 \mathbf{v}_t \right) \ , \tag{10}$$

derived from equations (8), as three independent equations for the six unknown components M_{ij} of the seismic moment; by multipling by n_i and summing over i, we get the first equation (8),

$$M_4 = M_{ij} n_i n_j = -4\pi\rho T R c_l^3 (\mathbf{v}_l \mathbf{n}) \tag{11}$$

which is not independent of the three equations written above. We view $\mathbf{v}_{l,t}$ as quantities measured experimentally and ρ, R, $c_{l,t}$ as known parameters; duration T will be determined shortly. We note the consistency (compatibility) relation $M_4^2 < M^2$, derived from $v_t^2 > 0$ ($v_{l,t}$ denote the magnitudes of the vectors $\mathbf{v}_{l,t}$). The inverse problem discussed in this paper is to determine the tensor M_{ij} from the displacement $\mathbf{v}_{l,t}$ in the far-field waves, making use of additional, model-related, information. The model we use is provided by the fault geometry of the focal zone.

Having known $\mathbf{M}$ and M_4 we can have access to the near-field diplacement given by equations (7), provided we know M_0.

Energy of earthquakes. If we multiply equation (3) by $\dot{u}_i$ and sum over the suffix i, we get the law of energy conservation

$$\begin{gathered} \frac{\partial}{\partial t} \left[\frac{1}{2}\rho \dot{u}_i^2 + \frac{1}{2}\rho c_t^2 (\partial_j u_i)^2 + \frac{1}{2}\rho (c_l^2 - c_t^2)(\partial_i u_i)^2 \right] - \\ -\rho c_t^2 \partial_j (\dot{u}_i \partial_j u_i) - \rho (c_l^2 - c_t^2) \partial_j (\dot{u}_j \partial_i u_i) = \dot{u}_i M_{ij}(t) \partial_j \delta(\mathbf{R}) \ ; \end{gathered} \tag{12}$$

according to this equation, the external force performs a mechanical work in the focus ($\dot{u}_i M_{ij}(t)\partial_j\delta(\mathbf{R})$ per unit volume and unit time); the corresponding energy is transferred to the waves (the term in the square brackets in equation (12)), which carry it through the space (the term including the *div* in equation (12)). It is worth noting that outside the focal region the force is vanishing; also, the waves do not exist inside the focal region; therefore, limiting ourselves to the displacement vector of the waves, we have not access to the mechanical work done by the external force in the focal region. This circumstance arises from the localized character of the focus.

In the far-field region we can use the decomposition $\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$ in longitudinal and transverse waves, where $curl\mathbf{u}_l = 0$ and $div\mathbf{u}_t = 0$; this decomposition leads to

$$\frac{\partial e_{l,t}}{\partial t} + c_{l,t} div \mathbf{s}_{l,t} = 0 \ \ , \tag{13}$$

where

$$e_{l,t}=\frac{1}{2}\rho\left(\dot{\mathbf{u}}_{l,t}^{f}\right)^{2}+\frac{1}{2}\rho c_{l,t}^{2}\left(\partial_{i}u_{l,tj}^{f}\right)^{2} \tag{14}$$

' is the energy density and

$$s_{l,ti}=-\rho c_{l,t}\dot{u}_{l,tj}^{f}\partial_{i}u_{l,tj}^{f} \tag{15}$$

are the components of the energy flux densities per unit time (the flow vectors). From equation (13) we can see that the energy is transported with velocities $c_{l,t}$. The volume energy $E=\int d\mathbf{R}(e_l+e_t)$ is equal to the total energy flux

$$\Phi=-\int dtd\mathbf{R}\left(c_{l}div\mathbf{s}_{l}+c_{t}div\mathbf{s}_{t}\right)=-\int dt\oint d\mathbf{S}\left(c_{l}\mathbf{s}_{l}+c_{t}\mathbf{s}_{t}\right)\ . \tag{16}$$

Making use of equations (8) and taking $h^{''}=-1/T^2$ we get

$$E=\Phi=\frac{4\pi\rho}{T}R^{2}\left(c_{l}v_{l}^{2}+c_{t}v_{t}^{2}\right)\ ; \tag{17}$$

this relation gives the energy released by the earthquake in terms of the displacement measured in the far-field region and the (short) duration of the earthquake. From equations (9) we get the relation

$$E=\frac{1}{4\pi\rho c_{t}^{5}T^{3}}\left[M^{2}-\left(1-c_{t}^{5}/c_{l}^{5}\right)M_{4}^{2}\right] \tag{18}$$

between energy and the seismic moment.

Geometry of the focal region. Let us consider a point torque $t_{ij}=f_ih_j$, where h_j are viewed as infinitesimal distances and f_i denote the components of a force $\mathbf{f}$; the force $\mathbf{f}$ originates in a volume force density $\partial_j\sigma_{ij}$, where σ_{ij} is the stress tensor; the latter can be expressed as $\sigma_{ij}=2\mu u_{ij}+\lambda u_{kk}\delta_{ij}$, where μ and λ are the Lame coefficients ($c_l^2=(2\mu+\lambda)/\rho$, $c_t^2=\mu/\rho$), $u_{ij}=\frac{1}{2}(\partial_j u_i+\partial_i u_j)$ is the strain tensor and $\mathbf{u}$, with components u_i, is the displacement vector.[13] We can write

$$\begin{gathered}t_{ij}=f_ih_j=\int d\mathbf{r}\partial_k\sigma_{ik}\cdot h_j=\\ =\mu\int d\mathbf{r}\partial_k^2u_i\cdot h_j+(\mu+\lambda)\int d\mathbf{r}\partial_k\partial_iu_k\cdot h_j=\\ =\mu\oint dS\cdot s_k\partial_ku_i\cdot h_j+(\mu+\lambda)\oint dS\cdot s_k\partial_iu_k\cdot h_j\ \ ,\end{gathered} \tag{19}$$

where the $\mathbf{r}$-integration is performed over the focal volume surrounded by the surface S and $\mathbf{s}$ is the unit vector of this surface. We may write $\partial_iu_k\simeq\Delta u_k/\Delta x_i$ for the derivatives of u_k and use $\frac{\Delta u_k}{\Delta x_i}\cdot h_j=\Delta u_k\delta_{ij}=u_k\delta_{ij}$, where u_k is the displacement on the surface. These equalities follow from the point-like nature of the torque. We note that $\mathbf{u}$ here is the focal displacement, which is distinct from the displacement in the waves. It folows

$$t_{ij}=\mu S\cdot\overline{s_ju_i}+(\mu+\lambda)S\cdot\overline{s_ku_k}\delta_{ij}\ \ , \tag{20}$$

where the overbar denotes the average over the surface with area S. This relation acquires a useful form for a localized (plane) fault. We assume that the fault focal region consists of two plane-parallel surfaces, each with (small) area S, separated by a small distance d, sliding against one another. The focal area is determined by two lengths $l_{1,2}$, $S=l_1l_2$. In general, the lengths l_1, l_2, d. In order to ensure the compatibility with the localization provided by the δ-function we assume $l_1=l_2=d=l$. In these conditions the product $\overline{s_iu_j}$ may be replaced by $2s_i\overline{u}_j$, where the vector $\mathbf{s}$ is the unit vector normal to the fault (we note that the integration over the surfaces perpendicular to the fault is zero, due to the opposing (sliding) displacements) . In view of the

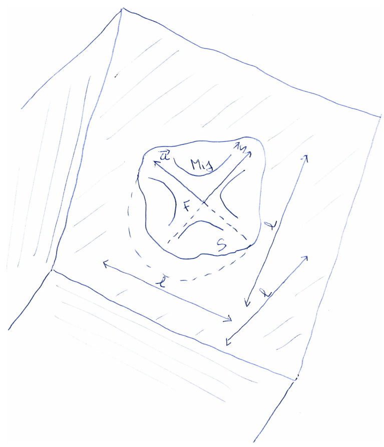

Figure 1: A fault focal cross-section with area S (dimension l, focus F); $\mathbf{s}$ is the unit vector normal to the fault and $\mathbf{a}$ is the unit vector of the focal displacement (in the plane of the fault); the seismic-moment tensor M_{ij} is represented by the rectangular hyperbola with the axes along the vectors $\mathbf{s}$ and $\mathbf{a}$.

small extension of the focal region, we may drop the average bar over u_j. In addition, this model of fault-slip implies $s_k u_k = 0$, *i.e.* the normal to the fault $\mathbf{s}$ and the focal displacement (fault slip) $\mathbf{u}$ are orthogonal vectors. In order to distinguish the focal displacement from the displacement in the seismic waves, we attach the superscript 0 to the focal displacement. The seismic moment is obtained by symmetrizing the expression given by equation (20); we get

$$M_{ij} = 2\mu S\left(s_i u_j^0 + s_j u_i^0\right) = 2\mu S u^0 \left(s_i a_j + a_i s_j\right) \quad , \tag{21}$$

where we introduce the unit vector $\mathbf{a}$ along the direction of the focal displacement; we write $u_i = u^0 a_i$, where u^0 is the magnitude of the focal displacement and $a_i^2 = 1$. We can see that the seismic moment is represented in equation (21) by two orthogonal vectors ($\mathbf{as} = 0$): the unit vector $\mathbf{a}$ along the focal displacement $\mathbf{u}^0$ and the unit vector $\mathbf{s}$, which gives the orientation of the focal surface. This is the moment-displacement relation derived by Kostrov[7, 8] for the slip along a (point-like) fault surface; it can be called a vectorial, or dyadic, representation of the seismic moment. We note the invariant $M_0 = M_{ii} = 0$, which tells that the seismic moment in this representation is a traceless tensor. This particularity gives access to the near-field waves (equations (7)), which become

$$\mathbf{u}_l^n = \frac{h_l}{8\pi\rho c_l^2 R^2}\left(4\mathbf{M} - 9M_4\mathbf{n}\right) \quad , \quad \mathbf{u}_t^n = -\frac{3h_t}{8\pi\rho c_t^2 R^2}\left(2\mathbf{M} - 3M_4\mathbf{n}\right) \tag{22}$$

($\mathbf{M}$ and M_4 are given by equations (10) and (11)). In addition, we note the relations $M_4^0 = M_{ij}s_i s_j = 0$ and $M_i^0 = M_{ij}s_j = 2\mu S u^0 a_i$; the former relation shows that the quadratic form associated to the seismic moment in the focal region is degenerate (it is represented by a conic), while the latter relation shows that the "force" in the focal region is directed along the focal displacement; both relations are expected from the Kostrov construction of the tensor of the fault seismic moment.

It is worth noting an uncertainty (indeterminacy) of the dyadic construction of the seismic-moment tensor. We can see from equation (21) that the seismic moment is invariant under the inter-change $\mathbf{s} \longleftrightarrow \mathbf{a}$. This means that from the knowledge of the seismic moment M_{ij} we cannot distinguish between the two orthogonal vectors $\mathbf{s}$ and $\mathbf{a}$ (fault direction and fault slip). Another symmetry of the seismic moment given by equation (21) is $\mathbf{s} \longleftrightarrow -\mathbf{a}$ (and $\mathbf{s} \longleftrightarrow -\mathbf{s}$, $\mathbf{a} \longleftrightarrow -\mathbf{a}$), which means that we cannot distinguish between the signs of the vectors $\mathbf{s}$ and $\mathbf{a}$ (as expected from the construction of the seismic moment in equation (21)).

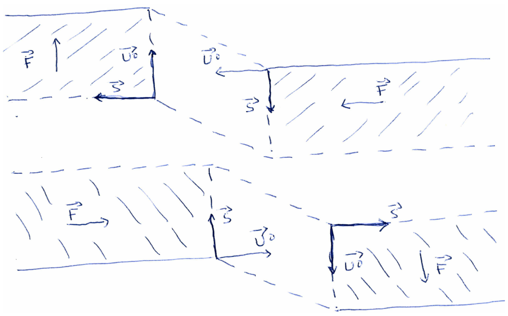

Figure 2: Two couples of sliding displacements ($\mathbf{u}^0$) and two orthogonal orientations ($\mathbf{s}$) in a fault focal region, illustrating the indeterminacy in the Kostrovconstruction of he seismic moemnt; F's denote the forces which give the torque.

In equation (21) the seismic moment is determined by four parameters: three components of the displacement vector $\mathbf{u}^0$ and one component of the (transverse) unit vector $\mathbf{s}$. By using this vectorial representation, the number of independent parameters of the seismic moment is reduced from six to four. We have, up to this moment, only the three equations (10) for these unknown parameters. The considerations made above for the vectorial representation of the seismic moment provides the fourth equation, relating the mechanical work W done in the focal region to the magnitude of the focal diplacement.

Indeed, from equation (12) the mechanical work in the focal region is given by

$$W = \int dt \int d\mathbf{R} \dot{u}_i^0(t) M_{ij}(t) \partial_j \delta(\mathbf{R}) \ ; \tag{23}$$

we may assume $\dot{u}_i^0(t) = \dot{h}(t) u_i^0$, and, since $M_{ij}(t) = M_{ij} h(t)$, we get

$$W = \frac{1}{2} \int d\mathbf{R} u_i^0 M_{ij} \partial_j \delta(\mathbf{R}) \ . \tag{24}$$

In this equation we may view the function $\delta(\mathbf{R})$ as corresponding to the shape of the focal surface, such that we may replace $\partial_j \delta(\mathbf{R})$ by s_j/l^4; using $V = l^3$ for the focal volume, we get $W \simeq \frac{1}{2l} u_i^0 M_{ij} s_j$; here, we may take approximately u^0 for l, which leads to $W \simeq \frac{1}{2} a_i M_{ij} s_j$. Therefore, making use of equation (21), we get $W \simeq \mu S u^0 = \mu V$; we can see that the mechanical work done in the focal region is of the order of the elastic energy stored in the focal region, as expected. By equating W with the energy E (and Φ) given by equation (17), the fourth equation

$$\mu V = \frac{4\pi\rho}{T} R^2 \left(c_l v_l^2 + c_t v_t^2 \right) \tag{25}$$

is obtained, which is the missing equation, needed for determining the seismic moment M_{ij} from the far-field seismic waves; it can also be written as

$$V = \frac{4\pi}{c_t^2 T} R^2 \left(c_l v_l^2 + c_t v_t^2 \right) \ . \tag{26}$$

This equation gives the volume of the focal region in terms of the displacement in the far-field seismic waves (provided duration T is known); the seismic moment given by equation (21) can be written as

$$M_{ij} = 2\mu V \left(s_i a_j + a_i s_j \right) \ , \tag{27}$$

where V can be inserted from equation (26). It remains to determine the vectors $\mathbf{a}$ and $\mathbf{s}$ from equations (10) in order to solve completely the inverse problem.

We note here the representation

$$u^0_{ij} = \frac{1}{2}\left(s_i a_j + a_i s_j\right) = \frac{1}{4\mu V} M_{ij} \tag{28}$$

for the focal strain, which follows immediately from the considerations made above on the geometry of the focal region; this equation relates the focal strain to the seismic moment.

We note that the estimations made above are affected by an order-of magnitude error in the numerical factors.

Solution of the inverse problem. Making use of the reduced moment $m_{ij} = M_{ij}/2\mu V$ and $m_i = M_i/2\mu V = M_{ij}n_j/2\mu V$, equation (21) leads to

$$s_i(\mathbf{na}) + a_i(\mathbf{ns}) = m_i \ ; \tag{29}$$

using equations (10) and (25) the components m_i of the reduced moment are given by

$$m_i = -\frac{T^2}{2R} \cdot \frac{c_l^3 v_{li} + c_t^3 v_{ti}}{c_l v_l^2 + c_t v_t^2} \ . \tag{30}$$

We solve here the equations (29) for the unit vectors **a** and **s**, subject to the conditions

$$s_i^2 = a_i^2 = 1 \ , \ s_i a_i = 0 \ . \tag{31}$$

Since $M_0 = 0$ and $M^2 > M_4^2$, we have $m_0 = m_{ii} = 0$ and $m^2 > m_4^2$ (where $m_4 = m_{ij}n_i n_j$ and $m^2 = m_i^2$). From equation (30) we have $m_i < 0$; the compatibility condition $m^2 > m_4^2$ can be checked immediately from equation (30) (it arises from $v_t^2 > 0$). We write equations (29) as

$$\alpha\mathbf{s} + \beta\mathbf{a} = \mathbf{m} \ , \tag{32}$$

where we introduce two new notations $\alpha = (\mathbf{na})$ and $\beta = (\mathbf{ns})$; also, we have

$$\beta\mathbf{s} + \alpha\mathbf{a} = \mathbf{n} \ . \tag{33}$$

From these two equations we get

$$2\alpha\beta = m_4 \ , \ \alpha^2 + \beta^2 = m^2 = 1 \ . \tag{34}$$

The equality $m^2 = 1$ has important consequences; it means $M^2 = (2\mu V)^2$, such that we can write the seismic moment from equation (27) as

$$M_{ij} = M\left(s_i a_j + a_i s_j\right) \ ; \tag{35}$$

it follows the magnitude of the seismic moment $\left({M_{ij}}^2\right)^{1/2} = \sqrt{2}M$; M is the magnitude of the projection of the seismic-moment tensor along the observation radius; in addition, from $E = W = \mu V$ we have $E = M/2 = \left({M_{ij}}^2\right)^{1/2}/2\sqrt{2}$. The magnitude $\left({M_{ij}}^2\right)^{1/2} = \sqrt{2}M = 2\sqrt{2}E$ may be used in the Gutenberg-Richter relation $\lg\left({M_{ij}}^2\right)^{1/2} = 1.5M_w + 16.1$, which defines the magnitude M_w of the earthquake; in terms of the earthquake energy this relation becomes $\lg E = 1.5(M_w - \lg 2) + 16.1$ (where $\lg 2 \simeq 0.3$). Further, from equation (30), the equality $m^2 = 1$ can be written as

$$\frac{T^4}{4R^2} \cdot \frac{c_l^6 v_l^2 + c_t^6 v_t^2}{\left(c_l v_l^2 + c_t v_t^2\right)^2} = 1 \ , \tag{36}$$

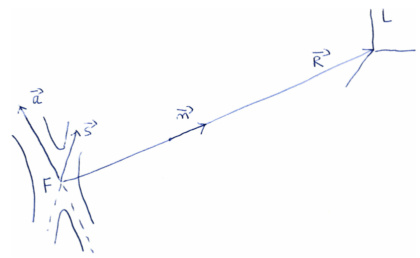

Figure 3: The hyperbola of the displacement ($\mathbf{a}$) in the fault plane (fault direction $\mathbf{s}$) at the focus ($\mathbf{F}$), seen from he local frame L.

which gives the duration T in terms of the displacements $v_{l,t}$ measured at distance R. Inserting T in equation (26), we get

$$V^2 = \frac{8\pi^2 R^3}{c_t^4}\left(c_l v_l^2 + c_t v_t^2\right)\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/2} \tag{37}$$

and the magnitude of the seismic moment and the energy of the earthquake

$$M = 2E = 2\mu V = 4\sqrt{2}\pi\rho R^{3/2}\left(c_l v_l^2 + c_t v_t^2\right)^{1/2}\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/4} \quad . \tag{38}$$

in terms of the displacements $v_{l,t}$ measured at distance R. In addition, eliminating R^2 between equations (26) and (36) we express the focal volume as

$$V = \frac{\pi T^3}{c_t^2}\cdot\frac{c_l^6 v_l^2 + c_t^6 v_t^2}{c_l v_l^2 + c_t v_t^2} \quad . \tag{39}$$

The solutions of the system of equations (34) are given by

$$\alpha = \sqrt{\frac{1+\sqrt{1-m_4^2}}{2}} \ , \ \ \beta = sgn(m_4)\sqrt{\frac{1-\sqrt{1-m_4^2}}{2}} \tag{40}$$

and $\alpha \longleftrightarrow \pm\beta$, α, $\beta \longleftrightarrow -\alpha$, $-\beta$. Making use of equations (30) and (36), the parameters m_i and m_4 are given by

$$m_i = -\frac{c_l^3 v_{li} + c_t^3 v_{ti}}{\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/2}} \ , \ \ m_4 = -\frac{c_l^3 \mathbf{v}_l \mathbf{n}}{\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/2}} \quad . \tag{41}$$

Finally, we get the vectors

$$\begin{gathered} \mathbf{s} = \frac{\alpha}{\alpha^2-\beta^2}\mathbf{m} - \frac{\beta}{\alpha^2-\beta^2}\mathbf{n} \ , \\ \mathbf{a} = -\frac{\beta}{\alpha^2-\beta^2}\mathbf{m} + \frac{\alpha}{\alpha^2-\beta^2}\mathbf{n} \ ; \end{gathered} \tag{42}$$

from equations (32) and (33); these solutions are symmetric under the operations $\mathbf{s} \longleftrightarrow \mathbf{a}$ ($\alpha \longleftrightarrow -\beta$) and $\mathbf{s} \longleftrightarrow -\mathbf{a}$ ($\alpha \longleftrightarrow \beta$, or α, $\beta \longleftrightarrow -\alpha$, $-\beta$). The seismic moment given by equation (35) is determined up to these symmetry operations.

The eigenvalues of the seismic moment given by equation (35) are $\pm M$; the corresponding eigenvectors $\mathbf{w}$ are given by $\mathbf{aw} = \pm\mathbf{sw}$, which imply $\mathbf{mw} = \pm\mathbf{nw}$; the vectors $\mathbf{w}$ are directed along the bisectrices of the angles made by $\mathbf{s}$ and $\mathbf{a}$, or $\mathbf{m}$ and $\mathbf{n}$. The associated quadratic form $M_{ij}x_i x_j = const$ is a rectangular hyperbola in the reference frame defined by the vectors $\mathbf{s}$ and $\mathbf{a}$; using coordinates $u = \mathbf{sx}$ and $v = \mathbf{ax}$, the equation of this hyperbola is $uv = const$. Making use of equations (35) and (42), this quadratic form can be written as

$$m_4\left(\xi^2 + \eta^2\right) - 2\xi\eta = const \quad , \tag{43}$$

where the coordinates $\xi = m_i x_i$ and $\eta = n_i x_i$ are directed along the vectors $\mathbf{m}$ and $\mathbf{n}$, respectively. The asymptotics of this hyperbola are $\xi = m_4\eta/\left(1+\sqrt{1-m_4^2}\right)$ and $\eta = m_4\xi/\left(1+\sqrt{1-m_4^2}\right)$ $(u = (\alpha\xi - \beta\eta)/(\alpha^2-\beta^2) = 0$ and $v = (-\beta\xi + \alpha\eta)/(\alpha^2-\beta^2) = 0)$.

The final solution for the seismic moment is

$$M_{ij} = \frac{M}{1-m_4^2}\left[m_i n_j + m_j n_i - m_4\left(m_i m_j + n_i n_j\right)\right] \quad , \tag{44}$$

where M is given by equation (38) and m_i, m_4 are given by equations (41); the focal strain is $u_{ij}^0 = M_{ij}/2M$.

Isotropic seismic moment. An isotropic seismic moment $M_{ij} = -M\delta_{ij}$ is an interesting particular case, since it can be associated with seismic events caused by explosions. In this case the transverse displacement is vanishing ($\mathbf{u}_t^{n,f} = 0$), $\mathbf{M} = -M\mathbf{n}$, $M_4 = -M$ and $\mathbf{v}_l = (R/c_l T)\mathbf{u}_l^n$ (equations (7) and (9)); from equations (10) and (17) we get

$$\mathbf{M} = -4\pi\rho T R c_l^3 \mathbf{v}_l \ , \ \ E = \frac{4\pi\rho R^2}{T} c_l v_l^2 \tag{45}$$

we can see that $\mathbf{v}_l\mathbf{n} > 0$ corresponds to $M > 0$ (explosion), while the (unphysical) case $\mathbf{v}_l\mathbf{n} < 0$ corresponds to an implosion. The focal zone is a sphere with radius of the order l, and the vectors $\mathbf{s}$ and $\mathbf{a}$ are both equal to the unit vector $\mathbf{n}$ ($\mathbf{s} = \mathbf{a} = \mathbf{n}$); the magnitude of the focal displacement is $u^0 = l$. The considerations made above for the geometry of the focal region lead to the representation

$$M_{ij} = -2V(2\mu+\lambda)\delta_{ij} = -2\rho c_l^2 V\delta_{ij} \quad , \tag{46}$$

where $V = Sl$ denotes the focal volume and S is the area of the focal region (we note that t_{ij} changes sign in equation (20)). Similarly, the energy is $E = W = \frac{1}{2}M$ ($M > 0$), such that, making use of equations (45), we get $c_l T = \sqrt{2Rv_l}$,

$$M = 2\pi\rho c_l^2 (2Rv_l)^{3/2} = 2\rho c_l^2 V \ , \tag{47}$$

and the focal volume $V = \pi(2Rv_l)^{3/2}$.

Discussion and concluding remarks. It is convenient to have an estimation of the order of magnitude of the various quantities introduced above. To this end we use a generic velocity c of the seismic waves and a generic vector $\mathbf{v}$ of the displacement in the far-field seismic waves. Equation (36) (which is equation $m^2 = 1$) gives $cT \simeq \sqrt{Rv}$, which provides an estimate of the duration of the earthquake in terms of the displacement measured at distance R. The focal volume can be estimated from equation (26) as $V \simeq (Rv)^{3/2} \simeq (cT)^3$, as expected (dimension l of the focal region of the order cT; the rate of the focal slip is $l/T \simeq c$). Also, from equation (38) we have the energy $E \simeq \mu V \simeq M \simeq \rho c^2 V$, where M is an estimate of the magnitude $\left(M_{ij}^2\right)^{1/2}$ of the seismic moment (and the magnitude of the vector $M_{ij}n_j$). From equation (28) we get a focal strain of the order unity, as expected.

In conclusion, it is shown in this paper that the displacement in the far-field seismic waves provides information about the structure of the focal region; in particular this displacement can be employed to determine the seismic-moment tensor for a fault slip, localized both in space and time (the inverse problem in Seismology). In this case the vectorial (Kostrov) representation of the seismic moment (dyadic representation) is written only with four (unknown) parameters; one is the magnitude of the focal displacement, while the other three define the spatial orientation of the seismic tensor (orientation of the fault). These unknown parameters are determined from the three equations relating the far-field displacement to the seismic tensor and the equation which relates the energy released in the earthquake (and carried by the seismic waves) to the focal displacement (and the fault focal volume). The solution of the resulting system of equations makes the graphical representation of the quadratic form associated to the seismic-moment tensor, which is a hyperbola, to offer a (three-dimensional) image of the focal region. The asymptotics of the hyperbola give the direction of the focal displacement and the orientation of the fault. Besides solving the inverse problem in Seismology for a localized fault slip, the geometry of the fault focal region (which leads to Kostrov representation) and the displacement in the far-field seismic waves provide reasonable estimations of the fault focal volume, focal strain, duration and energy of the earthquake and magnitude of the seismic moment. Also, the special case of an isotropic seismic moment is presented.

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