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Analysis of the cross-section of charge scattering by electromagnetic radiation

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Abstract

Additional information is given, concerning the scattering of non-relativistic charges by electromagnetic radiation confined in the focal region of the laser beams, which is relevant for the experimental investigation. Scattering angles, separation angles and momentum transfer are computed and multiple-photon emission and absorption regions are identified. The cross-section is estimated and the region of high momentum transfer is discussed. Particular cases of transverse scattering and longitudinal "scattering" are presented, as well as the elliptical polarization of the radiation. The effect caused by the confined radiation on the scattering by an external potential is included.

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Key words: scattering, non-relativistic charged particles, electromagnetic radiation

The scattering of non-relativistic charges by electromagnetic radiation has been investigated in Ref. [1]. The charges envisaged in Ref. [1] are electrons or ions, and the electromagnetic radiation is confined to the focal region of the laser beam, which has a finite spatial extension d. This scattering process is distinct from the Kroll-Watson scattering,[2] where the charges are immersed in the laser radiation and are scattered by atomic or nuclear targets. Further details are given in the present paper, which may be useful for experiments aimed at checking the predictions of Ref. [1].

We assume optical lasers with photon frequency of the order $\omega = 10^{15}s^{-1}$ (wavelength λ of the order $1\mu m$) and the dimension d of the focal region of the order a few tens of photon wavelengths (*e.g.*, $d \simeq 3\cdot 10^{-3}cm$). In the non-relativistic approximation the electric field $\mathbf{E} = \mathbf{E}_0 \sin\omega t$ is purely oscillating in time (t), where $\mathbf{E}_0$ is its amplitude, for linear polarization (its spatial dependence may be neglected, as well as the radiation magnetic field). The non-relativistic approximation requires $\mid e \mid A_0/mc^2 \ll 1$, where e is the particle charge, m is the particle mass, $A_0 = cE_0/\omega$ is the amplitude of the vector potential and c $(= 3 \times 10^{10}cm/s)$ is the speed of light in vacuum. This condition imposes $E_0 \ll 10^8 esu$ for electrons and $E_0 \ll 10^{11}A(esu)$ for ions, where A is the mass number of the ion. The laser intensity $(I = cE_0^2/8\pi)$ is limited to $10^{18}w/cm^2$ for electrons and $10^{24}w/cm^2$ for ions.

The cross-section of charge scattering by electromagnetic radiation given in Ref. [1] is

$$d\sigma_n = \frac{p_{fn}}{p_i}\left|\frac{m}{2\pi\hbar^2}\int d\mathbf{r}\left[n\hbar\omega + V(\mathbf{r})\right]J_n(e\mathbf{r}\mathbf{E}_0/\hbar\omega)e^{-\frac{i}{\hbar}\mathbf{p}_n\mathbf{r}}\right|^2 d\Omega \quad , \tag{1}$$

where J_n is the Bessel function of (integral) order n, $\mathbf{p}_i$ and $\mathbf{p}_{fn}$ are the initial and final momenta of the projectile, $\mathbf{p}_n = \mathbf{p}_{fn} - \mathbf{p}_i$ is the momentum transfer, $V(\mathbf{r})$ denotes an external potential, $d\Omega$ is the scattering solid angle and $\hbar$ is the Planck constant. For brevity, we omit the suffix n of the cross-section ($d\sigma$), the final momentum ($\mathbf{p}_f$) and the momentum transfer ($\mathbf{p}$). The integral in equation (1) is performed over the spatial region of dimension d. The initial and final momenta are related by the law of energy conservation

$$p_f^2/2m = p_i^2/2m + n\hbar\omega \ ; \tag{2}$$

this relation indicates scattering processes associated with emission and absorbtion of multiple photons.

We neglect for the moment the external potential $V(\mathbf{r})$ in equation (1). The cross-section due solely to the radiation scattering can be written as

$$d\sigma = \frac{p_f}{p_i}\left(\frac{mn\omega}{2\pi\hbar}\right)^2 \mid K_n \mid^2 d\Omega \ \ , \tag{3}$$

where

$$K_n = \int d\mathbf{r} J_n(\mathbf{ar}) e^{-i\mathbf{qr}} \ \ , \ \ \mathbf{a} = e\mathbf{E}_0/\hbar\omega \ \ , \ \ \hbar\mathbf{q} = \mathbf{p} \ \ , \tag{4}$$

or

$$K_n = \int dx J_n(ax) e^{-iq_{\parallel}x} \int d\mathbf{r}_{\perp} e^{-i\mathbf{q}_{\perp}\mathbf{r}_{\perp}} \ \ , \tag{5}$$

where the vectors $\mathbf{r} = (x, \mathbf{r}_{\perp})$, $\mathbf{q} = (q_{\parallel}, \mathbf{q}_{\perp})$ have been decomposed along two directions, one parallel with $\mathbf{a}$ (components x and $q_{\parallel}$) and the other perpendicular to $\mathbf{a}$ (components $\mathbf{r}_{\perp}$ and $\mathbf{q}_{\perp}$). The integral with respect to $\mathbf{r}_{\perp}$ shows that $q_{\perp}$ is approximately of the order $1/d$ and the integral can be approximated by d^2. Within this approximation we can see that the momentum transfer reduces to $p = \hbar q_{\parallel}$, *i.e.* the momentum is transferred along the electric field $\mathbf{E}_0$ ($\mathbf{a}$). We omit the suffix $\parallel$ in $q_{\parallel}$ and writes $K_n d^2 = I_n$, where

$$I_n = \int dx J_n(ax) e^{-iqx} \ \ , \ \ \hbar q = p \ ; \tag{6}$$

since $\mid a \mid = \mid e \mid E_0/\hbar\omega \ll 1/d$ and $q = p/\hbar \ll 1/d$, we may extend the integration in this integral to the whole space. The restriction of the momentum transfer to the direction $\mathbf{E}_0$ implies a cylindrical geometry for the scattering; the integration over angle φ in equation (3) leads to formally replacing $p_f \sin\theta d\varphi$ by $\hbar/d$; therefore, the cross-section given by equation (3) becomes

$$d\sigma = \frac{\hbar d^3}{p_i}\left(\frac{mn\omega}{2\pi\hbar}\right)^2 \mid I_n \mid^2 d\theta \ . \tag{7}$$

We pass now to the momentum conservation law

$$\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i \ \ , \tag{8}$$

where $\mathbf{p}$ is directed along the electric field $\mathbf{E}_0$ and $p_f = R_n p_i$, $R_n = \sqrt{1 + n\hbar\omega/E_i}$ from the energy conservation law given by equation (2), $E_i = p_i^2/2m$ being the initial energy. Let us assume that the electric field $\mathbf{E}_0$ makes an angle α with the initial direction of the projectile; it is sufficient to assume $0 < \alpha < \pi/2$. We parametrize the vectors occurring in equation (8) by $\mathbf{p} = p(\cos\alpha, \sin\alpha)$, $\mathbf{p}_f = p_f(\cos\theta, \sin\theta)$ and $\mathbf{p}_i = (1, 0)$ and get from equation (8)

$$p\cos\alpha = p_f\cos\theta - p_i \ \ , \ \ p\sin\alpha = p_f\sin\theta \tag{9}$$

and

$$\sin(\alpha-\theta)=\frac{\sin\alpha}{R_n} \ . \tag{10}$$

The solutions of this latter equation are

$$\begin{aligned} \theta_n &= \alpha - \arcsin\left(\tfrac{\sin\alpha}{R_n}\right) \ , \\ \theta_n &= \alpha - \pi + \arcsin\left(\tfrac{\sin\alpha}{R_n}\right) \ ; \end{aligned} \tag{11}$$

we can see that the scattering angles take discrete values, like for a diffraction scattering; the angles in the first row of equation (11) range from 0 to α, while those in the second row are comprised between $\alpha-\pi$ and 0. The momentum transfer given by equations (9) is

$$\begin{aligned} p &= p_i \tfrac{n\hbar\omega/E_i}{\sqrt{\cos^2\alpha+n\hbar\omega/E_i}+\cos\alpha} \ , \ \ 0<\theta_n<\alpha \ , \\ p &= -p_i\left(\sqrt{\cos^2\alpha+n\hbar\omega/E_i}+\cos\alpha\right) \ , \ \ \alpha-\pi<\theta_n<0 \ . \end{aligned} \tag{12}$$

There exist a few special scattering angles, according with these equations. For $n\to\infty$ the scattering angles are α and $\alpha-\pi$, *i.e.* the scattering takes place along the electric field (in both directions), with an infinite momentum and energy transfer. For $n=0$ the scattering is elastic and it takes place at angles 0 and $2\alpha-\pi$; the momentum transfer is zero for $\theta_0=0$ and $-2p_i\cos\alpha$ for $\theta_0=2\alpha-\pi$. For $\theta_{n_0}=\alpha-\pi/2$, where n_0 is given by $R_{n_0}=\sin\alpha$ ($n_0=-(E_i/\hbar\omega)\cos^2\alpha$) the scattering takes place along the direction perpendicular to the electric, with the momentum transfer $-p_i\cos\alpha$. For $0<\theta_n<\alpha$ and $\alpha-\pi<\theta_n<2\alpha-\pi$ the scattering is associated with photon absorption ($n>0$), while for $2\alpha-\pi<\theta_n<0$ the scattering implies photon emission. All this information is shown in Fig. 1. The final momenta $\mathbf{p}_f$ can be obtained by a graphical method. In the plane made by the vectors $\mathbf{p}_i$ and $\mathbf{E}_0$ we draw a succession of concentric spheres with radii $p_{fn}=p_i\sqrt{1+n\hbar\omega/E_i}$, n integer; the points of the intersection of these spheres with the line parallel with $\mathbf{E}_0$ and passing through the point indicated by the vector $\mathbf{p}_i$ give the scattering momenta $\mathbf{p}_{fn}$.

In order to get an observable diffraction pattern the integer n should take many values in the vicinity of $n=0$ and n_0. (On the other hand, n must be limited, of course, to the maximum number of photons $N\simeq E_0^2d^3/8\pi\hbar\omega\simeq 4\times10^5E_0^2$ comprised in the scattering region). Therefore, we should impose the condition $\hbar\omega/E_i\ll1$. From equations (11) the angle separation is given by

$$\Delta\theta_n=\frac{\sin\alpha}{2R_n^2\sqrt{R_n^2-\sin^2\alpha}}\frac{\hbar\omega}{E_i} \ ; \tag{13}$$

we can see that for large the diffraction maxima coalesce. In the vicinity of $n=0$ the separation angle is given by

$$\Delta\theta_0\simeq\tan\alpha\cdot\frac{\hbar\omega}{2E_i} \ , \tag{14}$$

while the separation angle near n_0 it is

$$\Delta\theta_n\simeq\frac{\sqrt{\hbar\omega/E_i\Delta n}}{2\sin\alpha(1+\Delta n\hbar\omega/E_i\sin^2\alpha)} \ , \tag{15}$$

where $\Delta n=n-n_0>0$; we can see that the diffraction maxima are more separated in this region in comparison with region $n\simeq0$. In order to have a good resolution, the separation angle $\Delta\theta_0$

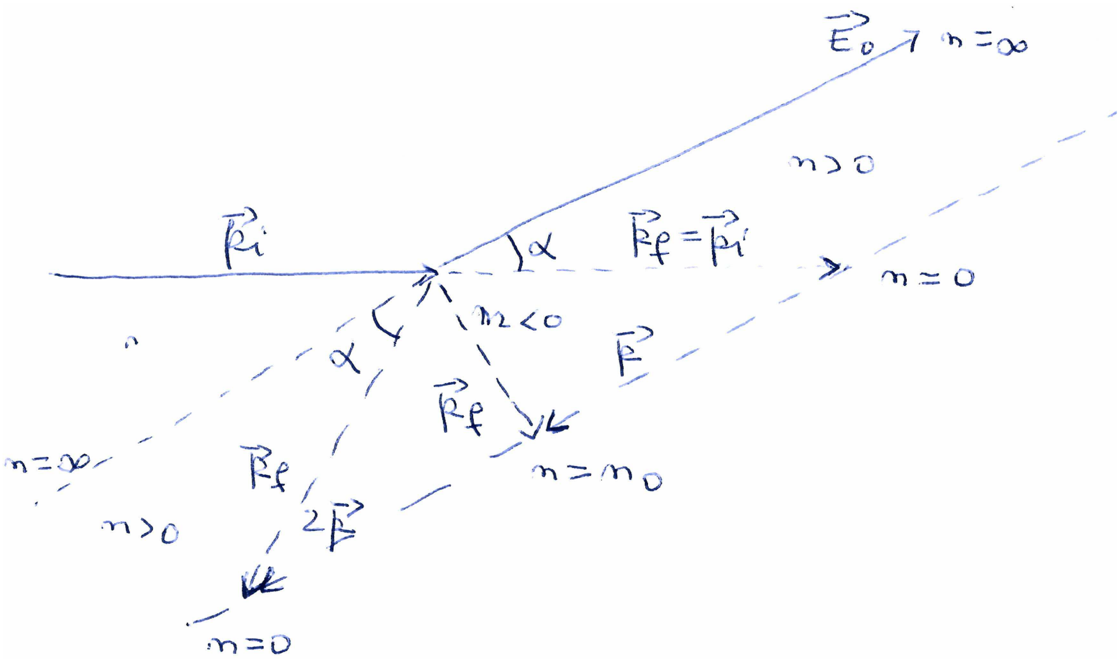

Figure 1: Special scattering angles for electric field $\mathbf{E}_0$, initial momentum $\mathbf{p}_i$, final momentum $\mathbf{p}_f$ and momentum transfer $\mathbf{p}$ and $2\mathbf{p}$; the regions with $n>0$ correspond to photon absorption, while the region with $n<0$ corresponds to photon emission. The line $n=n_0=-(E_i/\hbar\omega)\cos^2\alpha$, where E_i is the projectile energy and $\hbar\omega$ is the photon energy indicates the scattering angle $\theta=\alpha-\pi/2$.

should not be too small; for instance, a good resolution would be $\Delta\theta_0\simeq 10^{-3}$, which imposes E_i of the order $1keV$ (for $\hbar\omega=1eV$).

We turn now to the estimation of the cross-section given by equation (7). ‘

By using the generating function of the Bessel functions

$$e^{iz\cos\alpha}=\sum_n i^n J_n(z)e^{in\alpha}\ \ ,\ \ J_n(z)=\frac{(-i)^n}{2\pi}\int_{-\pi}^{\pi}d\alpha e^{iz\cos\alpha-in\alpha}\ \ , \tag{16}$$

we get

$$I_n=(-i)^n\int_{-\pi}^{\pi}d\alpha e^{-in\alpha}\delta(a\cos\alpha-q)\ \ . \tag{17}$$

The integral in equation (6) can be computed immediately, with the result

$$I_n=\frac{2(-i)^n}{\sqrt{a^2-q^2}}\cos n\alpha_0\ \ , \tag{18}$$

where $0<\alpha_0=\arccos(q/a)<\pi$ and $\mid q\mid<a$. For $q\ll a$, $\alpha_0\simeq\pi/2$ and $I_{2k}\simeq 2/a$ does not depend on k. The cross-section in this case is given by

$$d\sigma_{2k}\simeq\frac{\hbar d}{\pi^2 p_i}\left(\frac{\hbar\omega}{eE_0 d}\right)^2\left(\frac{\hbar\omega}{\hbar^2/2md^2}\right)^2 k^2 d\theta\ \ . \tag{19}$$

This result is valid for scattering angles $\theta\simeq 0$ (forward scattering), where the momentum transfer is small. With the numerical data used here and $E_i=1keV$, $E_0=10^8 esu$ the differential cross-section given by equation (19) is of the order $\simeq 10^{-8}k^2$. For larger scattering angles odd-order diffraction maxima appear and the diffraction spots increase in intensity (the projectile acquires more momentum transfer along directions parallel with the electric field). In the emission region $n<0$ the momentum transfer is of the order p_i. For large momentum transfer the denominator in equation (18) may vanish; then, it should be replaced by $\sqrt{a/d}$, and the cross-section given by equation (19) acquires an additional factor of the order $(\mid e\mid E_0 d/\hbar\omega)$.

For $\mid q\mid>\mid a\mid$ the integrals given by equation (6) are zero. It follows that the momentum transfer given by equations (12) is limited to $\mid p\mid<\mid e\mid E_0/\omega$ (the diffraction spots disappears for $\mid p\mid<\mid e\mid E_0/\omega$, in particular they are absent in the limit $n\rightarrow\infty$). Indeed, in equation (6) we may limit ourselves to $n\geq 0$ and a, $q>0$. The integrals I_n reduce to

$$L_n=\left[1+(-1)^n\right]L_n^c+\left[1-(-1)^n\right]L_n^s\ \ , \tag{20}$$

where

$$L_n^c=\int_0^\infty dxJ_n(x)\cos\lambda x\ \ ,\ \ L_n^s=\int_0^\infty dxJ_n(x)\sin\lambda x \tag{21}$$

and $\lambda=q/a>1$; further on, by an integration by parts,

$$L_n^c=\frac{1}{2\lambda}\left(L_{n+1}^s-L_{n-1}^s\right) \tag{22}$$

for $n\geq 1$. The integral L_n^s is known[3]:

$$L_n^s=\frac{\cos(n\pi/2)}{\sqrt{\lambda^2-1}\left[\lambda+\sqrt{\lambda^2-1}\right]^n}\ \ ,\ \ \lambda>1\ . \tag{23}$$

We can see that the integrals L_n, $n\geq 0$ are zero for $\lambda>1$.

Two particular cases may be worth discussing. First, let us assume that the initial momentum $\mathbf{p}_i$ is perpendicular to the electric field $\mathbf{E}_0$ (transverse scattering, $\alpha=\pi/2$). The scattering angles are given in this case by $\cos\theta_n=1/R_n$, the momentum transfer is $p_n=p_i\tan\theta_n$ and the separation angles are $\Delta\theta_n=\sqrt{\hbar\omega/nE_i}/2R_n^2$ ($R_n=\sqrt{1+n\hbar\omega/E_i}$) for $n\geq 0$ (there exists only absorption). For $\alpha=0$ (longitudinal "scattering") there is no scatering, the motion takes place along the direction of the electric field $\mathbf{E}_0$; the final momenta are given by $p_{fn}=p_i\sqrt{1+n\hbar\omega/E_i}$, with absorption for $n>0$ and emission for $0<n<-E_i/\hbar\omega$ (providing $\hbar\omega<E_i$).

The contribution of theexternal potential $V(\mathbf{r})$ to the corss-section given by equation (1) ca be estimated by using the Fourier transform of the potential. The momentum transfer is not confined to the direction of $\mathbf{E}_0$ anymore. The diffraction maxima are placed on the circles resulting from the intersection of the spheres with radii p_{fn} with the plane perpendicular to the plane made by the vectors $\mathbf{p}_i$ and $\mathbf{E}_0$, paralell with the vector $\mathbf{E}_0$and passing through the end point of $\mathbf{p}_i$.

The electric field with elliptic polarization has two components $E_1\sin\omega t$ and $E_2\cos\omega t$ in the plane perpendicular to the propagation vector. The energy conservation reads $p_f^2/2m=p_i^2/2m+(n_1+n_2)\hbar\omega$ and the cross-section is given by

$$d\sigma=\frac{p_f}{p_i}\left|\frac{m}{2\pi\hbar^2}\int d\mathbf{r}[(n_1+n_2)\hbar\omega+V(\mathbf{r})]J_{n_1}(eE_1x/\hbar\omega)J_{n_2}(eE_2y/\hbar\omega)e^{-\frac{i}{\hbar}\mathbf{pr}}\right|\ \ , \tag{24}$$

where $n_{1,2}$ are integers. The diffraction maxima are placed on ellipses, which depend on the sum n_1+n_2, and have a variable intensity, depending on $n_{1,2}$.

The information given in this paper may be relevant for the experimental investigation of the scattering of non-relativistic charges by electromagnetic radiation confined in the focal region of the laser beams.

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