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Quasiclassical transition probabilities

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According to Quantum Mechanics the uncertainty in the position of a particle is $\Delta q > \hbar/\Delta p$, where $\hbar$ is Planck's constant and Δp is the uncertainty in momentum (Heisenberg's uncertainty relation). In the non-relativistic approximation we can make Δp very large, such that q may be measured accurately. The relativistic theory should impose a limit upon this accuracy. Indeed, in $E^2 = p^2c^2 + m^2c^4$ p is motion momentum, and m is the mass of the particle, E is its energy and c is the speed of light in vacuum; the particle has a rest momentum mc. The accuracy in p should be smaller than mc, such that, in the rest frame, $\Delta q > \hbar/mc$ for a relativistic particle. This means that we cannot give the particle a well-defined coordinate, such that we cannot define a wavefunction for it; we cannot have a well-defined, standard, relativistic Quantum Mechanics.[1] If the particle is in motion $\Delta q > c\hbar/E$, and we need a high energy to define the coordinate; in the ultra-relativistic case $\Delta q > \hbar/p = \lambda$, where λ is the particle wavelength. This restriction agrees with the definition of a quantum-mechanical particle by means of its wavefunction, but it does not agree with the standard quantum-mechanical use of the wavefunction for any value of the coordinate. In this respect Quantum Mechanics imposes a serious limitation upon the nature of the particles.

Similarly, the time can only be measured relativistically with the accuracy $\Delta t > \hbar/mc^2$. These relativistic limitations arise from the existence of the rest energy mc^2, the rest momentum mc and the limiting velocity c. We are in the situation to use Quantum Mechanics at relativistic velocities, *i.e.* to use the notion of quanta, but without the standard use of wavefunctions defined at any position and any moment. Indeed, the basic asset of the Quantum Mechanics is the existence of quanta,[2] but only for distances larger than their wavelength and for times longer than the period of their frequency $\omega = E/\hbar$. It follows that particles can only be viewed as quanta which appear and disappear, in interaction processes, without, necessarily, a determined time evolution. This conclusion coincides with the standpoint of both Quantum Field Theory and Statistical Physics. We may call it the quasiclassical standpoint, since it overlooks the wavelengths and wave periods.

It is worth noting that, although in non-relativistic Quantum Mechanis we have the Schrodinger equation which gives a well-determined time evolution, this is limited by Heisenberg's uncertainty relations; in the relativistic theory the uncertainties in position and time acquire, partially, a

[1]L. Landau, "On the Quantum Theory of Fields", in *Niels Bohr and the Development of Physics*, ed. W. Pauli, Pergamon Press (1955) and references therein; W. Heitler, *The Quantum Theory of Radiation*, Oxford University Press (1944); L. Landau and R. Peierls, "Erweiterung des Unbestimmtheitsprinzips fur die relativistische Quantentheorie", Z. Physik **69** 56 (1931); L. Landau and R. Peierls, "Quantenelektrodynamik im Konfigurationsraum", Z. Physik **62** 188 (1930).

[2]A. Einstein, "Uber einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt", Ann. Physik **17** 132 (1905).

precise meaning, which precludes the definition of a wavefunction and its evolution equations. This can be seen most directly by the impossibility of defining a conserved probability. It is true that for fermions the spinors allow a probability, but this is done at the price of introducing the antiparticles, *i.e.* particles with negative energy; by writing the field as a superposition of $a_{\mathbf{p}}e^{ipx} + b^+_{\mathbf{p}}e^{-ipx}$ with four-dimesional notations), where $a_{\mathbf{p}}$ are annihilation operators for particles with positive energy and $b^+_{\mathbf{p}}$are creation operators for particles with negative energy, we must admit that the vacuum is full of antiparticles, which causes serious difficulties with infinite quantities. The field equations satisfied by bosons do not allow probabilities.

It follows that we are left, at least in relativistic Quantum Mechanics, with describing quantum transitions in interaction processes.

Let us assume an interaction energy (hamiltonian) V. Let us assume a wavefunction

$$\psi = e^{2\pi i \frac{S}{h}} \quad , \tag{1}$$

where S is the mechanical action (and $h = 2\pi\hbar$). Obviously, this is a periodic function for $\Delta S = h \times integer$. It follows that the (minimal) uncertainty relations read $\Delta p \Delta q \simeq h$ and $\Delta t \Delta E \simeq h$ (the first-order variation defines the determined trajectory). They differ from the usual ones $\Delta p \Delta q \simeq \frac{1}{2}\hbar$, $\Delta t \Delta E \simeq \frac{1}{2}\hbar$, because the latter are derived by assuming the wavefunction defined for any time and position; while, we see that it is meaningful only for periodicity $\Delta S = h \times integer$. From equation (1) we get the variation

$$\delta\psi = 2\pi i \frac{\delta S}{h}\psi \tag{2}$$

and the relative change in the number of quanta

$$\frac{|\delta\psi|^2}{|\psi|^2} = (2\pi)^2 \frac{|\delta S|^2}{h^2} \quad . \tag{3}$$

Since the change in action, caused by the interaction V during the small duration τ, is $\delta S = -\tau\delta E = -\tau V$, we get the relative change in the number of quanta per unit time

$$w = (2\pi)^2 \frac{\tau |V|^2}{h^2} \quad ; \tag{4}$$

this is the transition probability per unit time. According to the uncertainty relations we have $\tau = h/\Delta E$, where ΔE may be replaced by $\delta(\Delta E)$, in order to account for the energy conservation; we get

$$w = (2\pi)^2 \frac{|V|^2}{h} \delta(\Delta E) = \frac{2\pi}{\hbar} |V|^2 \delta(\Delta E) \quad . \tag{5}$$

The transition is performed by absorption and emission of quanta; $\delta(\Delta E)$ should account for the energy conservation of this process. Moreover, the energy V involved in this process is the matrix element V_{fi} between the final and initial states. Equation (5) becomes the standard formula for the transition probability per unit time (Fermi's golden rule). The multiplication by the probability of the states is necessary.

According to its definition w is $1/\tau$, where τ is the transition time; from equation (4) we have $\tau = \hbar / \mid V \mid$ and $w = \mid V \mid /\hbar$. This is valid as long as $\mid V \mid$ is much smaller than the energies $E_{i,f}$, $\hbar\omega$ involved in transition, where $\hbar\omega$ is the energy quantum which is responsible for transition. Indeed, this condition is the condition of validity of the perturbation theory. It amounts to say that the transition time τ should be much longer than the periods of the motions with energies

$E_{i,f}$, $\hbar\omega$. Indeed, in order to have a transition we need first to have well-defined states involved in that transition. On the other hand we use the approximation $\tau = \hbar/\Delta E = \hbar\delta(\Delta E)$, which amounts to $1/\mid V \mid = \delta(\Delta E)$; this equality tells that the energy is conserved in transition with the accuracy $\mid V \mid$. Therefore the transition rate is $\mid V \mid /\hbar$ and the energy conservation is achieved with the accuracy $\mid V \mid$.